Centro-Affine hypersurfaces with an induced almost paracontact structure

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Introduction

1. Introduction
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In the paper „$\tilde{J}$-tangent affine hypersurfaces with an induced almost paracontact structure” (submitted) I studied affine hypersurfaces $f: M \to \mathbb{R}^{2n+2}$ with an arbitrary $\tilde{J}$-tangent transversal vector field, where $\tilde{J}$ is the canonical paracomplex structure on $\mathbb{R}^{2n+2}$. Such a vector field induces in a natural way an almost paracontact structure $(\varphi, \xi, \eta)$ as well as the second fundamental form $h$. It was proved that if $(\varphi, \xi, \eta, h)$ is an almost paracontact metric structure then it is a para $\alpha$-Sasakian structure with $\alpha = -1$. Moreover, the hypersurface must be a piece of a hyperquadric.
Let $f : M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable $n$-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection $D$. Then for any transversal vector field $C$ we have

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C$$

(Gauss’ formula)

and

$$D_X C = -f_*(S X) + \tau(X)C,$$

(Weingarten’s formula)

where $X, Y$ are vector fields tangent to $M$. Here

- $\nabla$ — torsion free connection called the induced connection,
- $h$ — tensor of type (0,2) called the second fundamental form,
- $S$ — tensor of type (1,1) called the shape operator,
- $\tau$ — 1-form called the transversal connection form.
We have the following

**Fundamental equations, [Nomizu, Sasaki]**

For an arbitrary transversal vector field $C$ the induced connection $\nabla$, the second fundamental form $h$, the shape operator $S$, and the 1-form $\tau$ satisfy the following equations:

\begin{align*}
  R(X, Y)Z &= h(Y, Z)SX - h(X, Z)SY, \\
  (\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) &= (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \\
  (\nabla_X S)(Y) - \tau(X)SY &= (\nabla_Y S)(X) - \tau(Y)SX, \\
  h(X, SY) - h(SX, Y) &= 2d\tau(X, Y).
\end{align*}
Let $o$ be a point of the affine space $\mathbb{R}^{n+1}$ chosen as origin. An immersion $f$ of an $n$-manifold $M$ into $\mathbb{R}^{n+1} \setminus \{o\}$ such that $C = -of(x)$ for every $x \in M$ is always transversal to $f_*TM$ is called centro-affine hypersurface.
Blaschke hypersurface

We say that $f$ is **nondegenerate** if the second fundamental form $h$ is nondegenerate.

For a nondegenerate (orientable) hypersurface there exists a (global) transversal vector field $C$ satisfying the conditions:

$$\nabla \theta = 0, \quad \theta = \omega_h,$$

where $\omega_h$ is a volume element determined by $h$

$$\omega_h(X_1, \ldots, X_n) := \sqrt{|\det[h(X_i, X_j)]_{i,j=1\ldots n}|}$$

and $\theta$ is an induced volume element on $M$

$$\theta(X_1, \ldots, X_n) := \det[f_*X_1, \ldots, f_*X_n, C].$$

A transversal vector field satisfying these conditions is called **the affine normal field** or **the Blaschke normal field**. It is unique up to sign. A hypersurface with the transversal Blaschke normal field is called **the Blaschke hypersurface**.
A Blaschke hypersurface is called \textit{an affine hypersphere} if $S = \lambda I$, where $\lambda = \text{const}$.

If $\lambda = 0$, $f$ is called \textit{an improper affine hypersphere}, if $\lambda \neq 0$, $f$ is called \textit{a proper affine hypersphere}.
Affine hypersurfaces with a $\tilde{J}$-tangent transversal vector field

From now on we are interested in $(2n + 1)$-dimensional hypersurfaces $f : M \mapsto \mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2}$ is endowed with the standard paracomplex structure $\tilde{J}$, that is

$$\tilde{J}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_{n+1}, x_1, \ldots, x_{n+1}).$$

**Definition 1.**

A transversal vector field $C$ will be called $\tilde{J}$-tangent, if $\tilde{J}C \in f_*(TM)$. 

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The biggest \( \tilde{J} \) invariant distribution on \( M \) we denote by \( \mathcal{D} \). That is
\[
\mathcal{D}_x = f_*^{-1}(f_*(T_x M) \cap \tilde{J}(f_*(T_x M)))
\]
for every \( x \in M \). We have that \( \dim \mathcal{D}_x \geq 2n \). If for some \( x \) the \( \dim \mathcal{D}_x = 2n + 1 \) then \( \mathcal{D}_x = T_x M \) and it is not possible to find \( \tilde{J} \)-tangent transversal vector field in a neighbourhood of \( x \). Since we study only hypersurfaces with a \( \tilde{J} \)-tangent transversal vector field we always have \( \dim \mathcal{D} = 2n \). The distribution \( \mathcal{D} \) is smooth, since \( \dim \mathcal{D} \) is constant and is an intersection of two smooth distributions.

A vector field \( X \) is called a \( \mathcal{D} \)-field if \( X_x \in \mathcal{D}_x \) for every \( x \in M \). We use the notation \( X \in \mathcal{D} \) for vectors as well as for \( \mathcal{D} \)-fields.
Almost paracontact structures

A \((2n + 1)\)-dimensional manifold \(M\) is said to have an \textit{almost paracontact structure} if there exist on \(M\) a tensor field \(\varphi\) of type \((1,1)\), a vector field \(\xi\) and a 1-form \(\eta\) which satisfy

\[
\varphi^2(X) = X - \eta(X)\xi, \quad (5)
\]
\[
\eta(\xi) = 1 \quad (6)
\]

for every \(X \in TM\) and the tensor field \(\varphi\) induces an almost paracomplex structure on the distribution \(D = \ker \eta\), that is the eigendistributions \(D^+, D^-\) corresponding to the eigenvalues 1, \(-1\) of \(\varphi\) have equal dimension \(n\).
**Definition 2.**

Let \( f: M \to \mathbb{R}^{2n+2} \) (\( \dim M = 2n + 1 \)) be a hypersurface with a \( \tilde{J} \)-tangent transversal vector field \( C \). Then we define a vector field \( \xi \), a 1-form \( \eta \) and a tensor field \( \varphi \) of type (1,1) as follows:

\[
\xi := \tilde{J}C, \\
\eta|_D = 0 \text{ and } \eta(\xi) = 1, \\
\varphi|_D = \tilde{J}|_D \text{ and } \varphi(\xi) = 0.
\]

A structure \((\varphi, \xi, \eta)\) is called an **induced almost paracontact structure on** \( M \).
Theorem 1.

Let \( f : M \rightarrow \mathbb{R}^{2n+2} \) be an affine hypersurface with \( \tilde{J} \)-tangent transversal vector field \( C \). If \( (\varphi, \xi, \eta) \) is an induced almost paracontact structure on \( M \) then the following equations hold:

\[
\eta(\nabla_X Y) = h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X),
\]

(7)

\[
\varphi(\nabla_X Y) = \nabla_X \varphi Y - \eta(Y)SX - h(X, Y)\xi,
\]

(8)

\[
\eta([X, Y]) = h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X))
\]

\[
+ \eta(Y)\tau(X) - \eta(X)\tau(Y),
\]

(9)

\[
\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \eta(X)SY - \eta(Y)SX,
\]

(10)

\[
\eta(\nabla_X \xi) = \tau(X),
\]

(11)

\[
\eta(SX) = -h(X, \xi)
\]

(12)

for every \( X, Y \in \mathcal{X}(M) \).
Proof. For every $X \in TM$ we have
$$\tilde{J}X = \varphi X + \eta(X)C.$$ 
Furthermore
$$\tilde{J}(D_X Y) = \tilde{J}(\nabla_X Y + h(X, Y)C) = \tilde{J}(\nabla_X Y) + h(X, Y)\tilde{J}C$$
$$= \varphi(\nabla_X Y) + \eta(\nabla_X Y)C + h(X, Y)\xi$$
and
$$D_X \tilde{J}Y = D_X(\varphi Y + \eta(Y)C) = D_X\varphi Y + X(\eta(Y))C + \eta(Y)D_X C$$
$$= \nabla_X \varphi Y + h(X, \varphi Y)C + X(\eta(Y))C + \eta(Y)(-SX + \tau(X)C)$$
$$= \nabla_X \varphi Y - \eta(Y)SX + (h(X, \varphi Y) + X(\eta(Y) + \eta(Y)\tau(X)))C.$$ 
Since $D_X \tilde{J}Y = \tilde{J}(D_X Y)$, comparing transversal and tangent parts, we obtain (7) and (8) respectively. Equations (9)—(12) follow directly from (7) and (8).
Let $f: M \to \mathbb{R}^{n+2}$ be an immersion, and $\mathcal{N}: M \ni x \mapsto N_x$ be a transversal bundle for the immersion $f$. Immersion $f$ together with the transversal bundle $\mathcal{N}$ we call an affine hypersurface of codimension two
Let \( g : M^{2n} \to \mathbb{R}^{2n+2} \) be an immersion and let \( \tilde{J} \) be the standard paracomplex structure on \( \mathbb{R}^{2n+2} \). We always identify \((\mathbb{R}^{2n+2}, \tilde{J})\) with \( \tilde{\mathbb{C}}^{n+1} \), where \( \tilde{\mathbb{C}} \) is the real algebra of para-complex numbers. We assume that \( g^*(TM) \) is \( \tilde{J} \)-invariant and \( \tilde{J}|_{g^*(T_xM)} \) is a paracomplex structure on \( g^*(T_xM) \) for every \( x \in M \). Then \( J \) induces an almost paracomplex structure on \( M \) which we will also denote by \( \tilde{J} \). Moreover, since \((\mathbb{R}^{2n+2}, \tilde{J})\) is para-complex then \((M, \tilde{J})\) is para-complex as well. By assumption we have that \( dg \circ \tilde{J} = \tilde{J} \circ dg \) that is \( g : M^{2n} \to \mathbb{R}^{2n+2} \cong \tilde{\mathbb{C}}^{n+1} \) is a para-holomorphic immersion. Since para-complex dimension of \( M \) is \( n \), immersion \( g \) is called a para-holomorphic hypersurface.
Let $g : M^{2n} \to \mathbb{R}^{2n+2}$ be an affine hypersurface of codimension 2 with a transversal bundle $\mathcal{N}$.

If $g$ is para-holomorphic then it is called \textit{affine para-holomorphic hypersurface}. If additionally the transversal bundle $\mathcal{N}$ is $\tilde{J}$-invariant then $g$ is called a \textit{para-complex affine hypersurface}.
Definition 3.
Let $g : M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. We say that $g$ is para-complex centro-affine hypersurface if $\{g, \tilde{J}g\}$ is a transversal bundle for $g$.

Theorem 2.
Let $g : M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. Then for every $x \in M$ there exists a neighborhood $U$ of $x$ and a transversal vector field $\zeta : U \to \mathbb{R}^{2n+2}$ such that $\{\zeta, \tilde{J}\zeta\}$ is a transversal bundle for $g|_U$. That is $g|_U$ considered with $\{\zeta, J\zeta\}$ is a para-complex affine hypersurface.
**Proof.** Indeed, let $N_x$ be any vector space transversal to $g_*(T_xM)$. If $N_x$ is $\tilde{J}$-invariant then it must be a para-complex vector space so we can find vector $v \in N_x$ such that $\{v, \tilde{J}v\}$ is a basis for $N_x$. If $N_x$ is not $\tilde{J}$-invariant then $N_x \cap \tilde{J}N_x$ must be 1-dimensional. In this case we can choose $v \in N_x$ such that $v \not\in N_x \cap \tilde{J}N_x$. Now vector $\tilde{J}v$ is transversal to $g_*(T_xM)$ and linearly independent with $v$. That is $\{v, \tilde{J}v\}$ is a para-complex transversal vector space to $g_*(T_xM)$.

Summarizing at $x$ we can always find a transversal vector $v$ such that $g_*(T_xM) \oplus \text{span}\{v, \tilde{J}v\} = \mathbb{R}^{2n+2}$.

Hence in a neighborhood of $x$ we can find a transversal vector field $\zeta$ such that $\{\zeta, \tilde{J}\zeta\}$ is a transversal bundle for $g$ in this neighborhood.
Now, let \( g: M^{2n} \rightarrow \mathbb{R}^{2n+2} \) be a para-holomorphic hypersurface and let \( \zeta: U \rightarrow \mathbb{R}^{2n+2} \) be a local transversal vector field on \( U \subset M \) such that \( \{\zeta, \tilde{J}\zeta\} \) is a transversal bundle to \( g \).

So for all tangent vector fields \( X, Y \in \mathcal{X}(U) \) we can decompose \( D_X Y \) and \( D_X \zeta \) into tangent and transversal part.

That is we have

\[
D_X g^* Y = g^*(\nabla_X Y) + h_1(X, Y)\zeta + h_2(X, Y)\tilde{J}\zeta \quad \text{(formula of Gauss)}
\]

\[
D_X \zeta = -g^*(SX) + \tau_1(X)\zeta + \tau_2(X)\tilde{J}\zeta \quad \text{(formula of Weingarten)}
\]

where \( \nabla \) is a torsion free affine connection on \( U \), \( h_1 \) and \( h_2 \) are symmetric bilinear forms on \( U \), \( S \) is a \((1,1)\)-tensor field on \( U \) and \( \tau_1 \) and \( \tau_2 \) are 1-forms on \( U \).
Using the fact that $D\tilde{J} = 0$ and the formula of Gauss by straightforward computations we can prove the following

**Lemma 1.**

\[\nabla \tilde{J} = 0,\]
\[h_1(X, \tilde{J}Y) = h_1(\tilde{J}X, Y) = h_2(X, Y),\]
\[h_2(X, \tilde{J}Y) = h_1(X, Y).\]

We say that a hypersurface is *nondegenerate* if $h_1$ (and in consequence $h_2$) is nondegenerate.
Now assume that \( \{\tilde{\zeta}, \tilde{J}\tilde{\zeta}\} \) is any other transversal bundle on \( U \). Then there exist functions \( \varphi, \psi \) on \( U \) and \( Z \in \mathcal{X}(U) \) such that

\[
\tilde{\zeta} = \varphi \zeta + \psi \tilde{J}\zeta + g_* Z.
\]

Since \( \{\tilde{\zeta}, \tilde{J}\tilde{\zeta}\} \) is transversal the above formula implies that \( \varphi^2 - \psi^2 \neq 0 \). Indeed, we have

\[
\varphi \tilde{\zeta} - \psi \tilde{J}\tilde{\zeta} = (\varphi^2 - \psi^2)\zeta + \varphi g_* Z - \psi \tilde{J}g_* Z.
\]

If \( \varphi^2 - \psi^2 = 0 \) then \( \varphi \tilde{\zeta} - \psi \tilde{J}\tilde{\zeta} \in TU \), but since \( \{\tilde{\zeta}, \tilde{J}\tilde{\zeta}\} \) is transversal we obtain \( \varphi = \psi = 0 \) what is impossible because \( \tilde{\zeta} \) is transversal.
By the formulas of Gauss and Weingarten with respect to $\tilde{\zeta}$ we obtain the objects $\tilde{\nabla}, \tilde{h}_1, \tilde{h}_2, \tilde{S}, \tilde{\tau}_1, \tilde{\tau}_2$ which satisfy the following relations

### Lemma 2.

\begin{align*}
    h_1(X, Y) &= \varphi \tilde{h}_1(X, Y) + \psi \tilde{h}_2(X, Y), \\
    h_2(X, Y) &= \psi \tilde{h}_1(X, Y) + \varphi \tilde{h}_2(X, Y), \\
    \nabla_X Y &= \tilde{\nabla}_X Y + \tilde{h}_1(X, Y)Z + \tilde{h}_2(X, Y)\tilde{J}Z, \\
    \varphi S X + \psi S X - \nabla_X Z &= \tilde{S} X - \tilde{\tau}_1(X)Z - \tilde{\tau}_2(X)\tilde{J}Z, \\
    \varphi \tilde{\tau}_1(X) + \psi \tilde{\tau}_2(X) &= X(\varphi) + \varphi \tau_1(X) + \psi \tau_2(X) + h_1(X, Z), \\
    \psi \tilde{\tau}_1(X) + \varphi \tilde{\tau}_2(X) &= \varphi \tau_2(X) + X(\psi) + \psi \tau_1(X) + h_2(X, Z), \\
    \tilde{h}_1 &= \frac{h_1 \varphi - h_2 \psi}{\varphi^2 - \psi^2}, \\
    \tilde{\tau}_1(X) &= \frac{1}{2} X(\ln |\varphi^2 - \psi^2|) + \tau_1(X) \\
    &\quad + \frac{1}{\varphi^2 - \psi^2} (\varphi h_1(X, Z) - \psi h_2(X, Z)).
\end{align*}
Proof. Formulas (16) to (21) are straightforward. Formulas (22) and (23) follow at once from (16), (17), (20) and (21).
On $U$ we define the volume form $\theta_\zeta$ by the formula

$$\theta_\zeta(X_1, \ldots, X_{2n}) := \det(g_\ast X_1, \ldots, g_\ast X_{2n}, \zeta, \tilde{J}_\zeta)$$

for tangent vectors $X_i$, $i = 1, \ldots, 2n$. Then, consider the function $H_\zeta$ on $U$ defined by

$$H_\zeta := \det[h_1(X_i, X_j)]_{i,j=1\ldots 2n}$$

where $X_1, \ldots, X_{2n}$ is a local basis in $TU$ such that $\theta_\zeta(X_1, \ldots, X_{2n}) = 1$. This definition is independent of the choice of basis.

Moreover, we also have

$$\nabla X \theta_\zeta = 2\tau_1(X) \theta_\zeta.$$
If \( \{\tilde{\zeta}, \tilde{J}\tilde{\zeta}\} \) is other transversal bundle on \( U \) then we have the following relations between \( \theta_{\tilde{\zeta}}, H_{\tilde{\zeta}} \) and \( \theta_{\zeta}, H_{\zeta} \)

Lemma 3.

\[
\theta_{\tilde{\zeta}} = (\varphi^2 - \psi^2)\theta_{\zeta}, \quad (24)
\]

\[
H_{\tilde{\zeta}} = \frac{1}{(\varphi^2 - \psi^2)^{n+2}} \cdot H_{\zeta}. \quad (25)
\]
Proof. Formula (24) is straightforward. So, we only prove (25). Let \( \{X_1, \tilde{J}X_1, \ldots, X_n, \tilde{J}X_n\} \) be a local basis on \( TM \). Then
\[
\theta_\zeta(X_1, \tilde{J}X_1, \ldots, X_n, \tilde{J}X_n) = \alpha
\]
where \( \alpha \neq 0 \) (so either \( \alpha < 0 \) or \( \alpha > 0 \)). Now let \( \tilde{X}_1 := \frac{X_1}{\sqrt{|\alpha|}} \) then
\[
\theta_\zeta(\tilde{X}_1, \tilde{J}\tilde{X}_1, X_2, \tilde{J}X_2, \ldots, X_n, \tilde{J}X_n) = \frac{\alpha}{|\alpha|}.
\]
So we can choose the basis \( \{X_1, \tilde{J}X_1, \ldots, X_n, \tilde{J}X_n\} \) such that
\[
\theta_\zeta(X_1, \tilde{J}X_1, \ldots, X_n, \tilde{J}X_n) = \pm 1.
\]
Let \( Y_i = \frac{X_i}{|\varphi^2 - \psi^2|^{\frac{1}{2n}}} \) for \( i = 1, \ldots, n \). Then

\[
\theta_{\zeta}(Y_1, \ldots, \tilde{Y}_n) = (\varphi^2 - \psi^2)\theta_{\zeta}(Y_1, \ldots, \tilde{Y}_n)
= (\varphi^2 - \psi^2) \cdot \frac{1}{|\varphi^2 - \psi^2|} \theta_{\zeta}(X_1, \ldots, X_{2n})
= \text{sgn}(\varphi^2 - \psi^2)\theta_{\zeta}(X_1, \ldots, X_{2n}) = \pm 1,
\]

so

\[
H_{\zeta} = \det \left[ \tilde{h}_1(Y_i, Y_j) \right]
= \frac{1}{(\varphi^2 - \psi^2)^2} \det \left[ \tilde{h}_1(X_i, X_j) \right].
\]
We also compute
\[
\det \begin{bmatrix} \tilde{h}_1(X_k, X_l) & \tilde{h}_1(X_k, \tilde{J}X_l) \\ \tilde{h}_1(X_m, X_l) & \tilde{h}_1(X_m, \tilde{J}X_l) \end{bmatrix} = \frac{1}{\varphi^2 - \psi^2} \det \begin{bmatrix} h_1(X_k, X_l) & h_1(X_k, \tilde{J}X_l) \\ h_1(X_m, X_l) & h_1(X_m, \tilde{J}X_l) \end{bmatrix}.
\]

The above implies that
\[
H_{\tilde{\zeta}} = \frac{1}{(\varphi^2 - \psi^2)^2} \cdot \frac{1}{(\varphi^2 - \psi^2)^n} \cdot H_{\zeta}
\]
and eventually
\[
H_{\tilde{\zeta}} = \frac{1}{(\varphi^2 - \psi^2)^{n+2}} \cdot H_{\zeta}.
\]
Affine normal vector fields

Definition 4.

When $g$ is nondegenerate there exist transversal vector fields $\zeta$ satisfying the following two conditions:

$|H_\zeta| = 1,$

$\tau_1 = 0.$

Such vector fields are called **affine normal vector fields**.

**Proof.** Let $\{\zeta, \tilde{J}\zeta\}$ be an arbitrary transversal bundle for $g$. Since $g$ is nondegenerate we have $H_\zeta \neq 0$ so we can find functions $\varphi$ and $\psi$ such that $\varphi^2 - \psi^2 \neq 0$ and

$$|(\varphi^2 - \psi^2)^{n+2}| = |H_\zeta|.$$  \hfill (26)

Let $\tilde{\zeta} := \varphi \zeta + \psi \zeta + Z$ where $Z$ is an arbitrary vector field on $M$. Lemma 3 (formula (25)) and formula (26) imply that $|H_{\tilde{\zeta}}| = 1$. 

We shall show that we can choose $Z$ in such a way that $\tilde{\zeta}$ is an affine normal vector field.

By Lemma 2 (formula (23)) we have

$$\tilde{\tau}_1(X) = \frac{1}{2}X(\ln |\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2}(\varphi h_1(X, Z) - \psi h_2(X, Z))$$

Now using Lemma 1 we obtain

$$\tilde{\tau}_1(X) = \frac{1}{2}X(\ln |\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2} \cdot h_1(X, \varphi Z - \psi \tilde{J}Z).$$

Since $h_1$ is nondegenerate we can find $Z$ such that $\tilde{\tau}_1(X) = 0$ for all vector fields $X$ defined on $U$. In this way we have shown that on every para-holomorphic hypersurface one may find (at least locally) an affine normal vector field.
Lemma 4.

Let $g : M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be a nondegenerate para-holomorphic hypersurface and let $\zeta, \tilde{\zeta} : U \rightarrow \mathbb{R}^{2n+2}$ be two affine normal vector fields on $U \subset M$. Then $\tilde{\zeta} = \varphi \zeta + \psi \tilde{J}\zeta$ where $|\varphi^2 - \psi^2| = 1$.

Proof. Since $\zeta, \tilde{\zeta}$ are transversal there exist functions $\varphi, \psi \in C^\infty(U)$ and a tangent vector field $Z \in \mathcal{X}(U)$ such that $\tilde{\zeta} = \varphi \zeta + \psi \tilde{J}\zeta + Z$. Since $|H_\zeta| = |H_{\tilde{\zeta}}| = 1$ the formula (25) implies that $|\varphi^2 - \psi^2| = 1$. Now, due to the fact that $\tau_1 = \tilde{\tau}_1 = 0$ and by formulas (23) and Lemma 1 we obtain

$$0 = \varphi h_1(X, Z) - \psi h_2(X, Z) = \varphi h_1(X, Z) - \psi h_1(X, \tilde{J}Z) = h_1(X, \varphi Z - \psi \tilde{J}Z)$$

for all $X \in \mathcal{X}(U)$. Since $h_1$ is non-degenerate and $\varphi^2 - \psi^2 \neq 0$ the last formula implies that $Z = 0$. The proof is completed.
Definition 5.

A nondegenerate para-complex hypersurface is said to be a **proper para-complex affine hypersphere** if there exists an affine normal vector field \( \zeta \) such that \( S = \alpha I \), where \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( \tau_2 = 0 \).

If there exists an affine normal vector field \( \zeta \) such that \( S = 0 \) and \( \tau_2 = 0 \) we say about an **improper para-complex affine hypersphere**.
**Example 1** Let $g : \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x, y) := \frac{1}{2} \begin{pmatrix} \cos x \\ \sin x \\ \cos x \\ \sin x \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos y \\ \sin y \\ -\cos y \\ -\sin y \end{pmatrix}.$$  

(27)

It easily follows that $g$ is an immersion. Moreover $\tilde{J}g_x = g_x$ and $\tilde{J}g_y = -g_y$ so $g$ is a para-holomorphic hypersurface. If we take $\zeta := -g$ then $\{\zeta, \tilde{J}\zeta\}$ is a transversal bundle for $g$. By straightforward computations we obtain

\begin{align*}
h_1 &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad h_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad S = \text{id}, \quad \tau_1 = \tau_2 = 0
\end{align*}

relative to the canonical basis $\{\partial_x, \partial_y\}$. 
Moreover, since

\[ \theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \tilde{J}\zeta] = \frac{1}{2} \]

one may easily compute that \( H_\zeta = 1 \) that is \( g \) is a proper para-complex affine sphere.
Example 2 Let \( g : \mathbb{R}^2 \to \mathbb{R}^4 \) be given by the formula

\[
g(x, y) := \frac{1}{2} \begin{pmatrix} \cosh x \\ \sinh x \\ \cosh x \\ \sinh x \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cosh y \\ \sinh y \\ -\cosh y \\ -\sinh y \end{pmatrix}.
\]  

(28)

Exactly like in the previous example we have that \( g \) is an immersion and \( \tilde{J}g_x = g_x \) and \( \tilde{J}g_y = -g_y \) so \( g \) is a para-holomorphic hypersurface. Again taking \( \zeta := -g \) we obtain that \( \{\zeta, \tilde{J}\zeta\} \) is a transversal bundle for \( g \). We also have

\[
h_1 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad h_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad S = \text{id}, \quad \tau_1 = \tau_2 = 0
\]

relative to the canonical basis \( \{\partial_x, \partial_y\} \).
Moreover, since

\[ \theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \tilde{J}\zeta] = \frac{1}{2} \]

we easily compute that \( H_\zeta = 1 \) that is \( g \) is a proper para-complex affine sphere.
Example 3 Let \( g : \mathbb{R}^2 \to \mathbb{R}^4 \) be given by the formula

\[
g(x, y) := \frac{1}{2} \begin{pmatrix} \cosh x \\ \sinh x \\ - \cosh x \\ - \sinh x \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \cos y \\ \sin y \\ \cos y \\ \sin y \end{pmatrix}.
\]  

(29)

It easily follows that \( g \) is an immersion and \( \tilde{J}g_x = g_x \) and \( \tilde{J}g_y = -g_y \) so \( g \) is a para-holomorphic hypersurface. If we take \( \zeta := -g \) then \( \{ \zeta, \tilde{J}\zeta \} \) is a transversal bundle for \( g \).
Example 4 Let \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be given by the formula
\[
g(x, y) := \frac{1}{2} \begin{pmatrix} x \\ \frac{1}{2} x^2 \\ x \\ \frac{1}{2} x^2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} y \\ \frac{1}{2} y^2 \\ -y \\ -\frac{1}{2} y^2 \end{pmatrix}.
\] (30)

It easily follows that \( g \) is an immersion and \( \tilde{J}g_x = g_x \) and \( \tilde{J}g_y = -g_y \) so \( g \) is a para-holomorphic hypersurface. Let \( \zeta := (0, 0, 0, 1)^T \) then \( \tilde{J}\zeta = (0, 1, 0, 0)^T \) and \( \{\zeta, \tilde{J}\zeta\} \) is a transversal bundle for \( g \). We compute
\[
h_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad h_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad S = 0, \quad \tau_1 = \tau_2 = 0
\]
relative to the canonical basis \( \{\partial_x, \partial_y\} \).
Since

\[ \theta_{\zeta}(\partial_x, \partial_y) := \det [g_x, g_y, \zeta, \tilde{J}\zeta] = -\frac{1}{2} \]

then \( H_{\zeta} = -1 \) that is \( g \) is an improper para-complex affine sphere.
Lemma 5.

Let $F : I \rightarrow \mathbb{R}^{2n}$ be a smooth function on interval $I$. If $F$ satisfies the differential equation

$$F'(z) = -\tilde{J}F(z),$$

(31)

then $F$ is of the form

$$F(z) = \tilde{J}v \cosh z - v \sinh z,$$

(32)

where $v \in \mathbb{R}^{2n}$.

Proof. It is not difficult to check, that functions of the form (32) satisfy differential equation (31). On the other hand, since (31) is a first order ordinary differential equation, the Picard-Lindelöf theorem implies that any solution of (31) must be of the form (32).
Theorem 3.

Let \( f : M \to \mathbb{R}^{2n+2} \) be a centro-affine hypersurface with a \( \tilde{J} \)-tangent centro-affine vector field. Then there exist an open subset \( U \subset \mathbb{R}^{2n} \), an interval \( I \subset \mathbb{R} \) and an immersion \( g : U \to \mathbb{R}^{2n+2} \) such that \( f \) can be locally expressed in the form

\[
  f(x_1, \ldots, x_{2n}, z) = \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z
\]

for all \( (x_1, \ldots, x_{2n}, z) \in U \times I \).

Proof. Denote \( C := -f \). Since \( f \) is centro-affine hypersurface with \( \tilde{J} \)-tangent transversal vector field then we have \( \tilde{J}C = -\tilde{J}f \in f_*(TM) \). Therefore for every \( x \in M \) there exists a neighborhood \( V \) of \( x \) and a map \( \psi(x_1, \ldots, x_{2n}, z) \) on \( V \) such that

\[
f_* \frac{\partial}{\partial z} = \tilde{J}C.
\]
That is $f$ can be locally expressed in the form $f(x_1, \ldots, x_{2n}, z)$, where $f_z = -\tilde{J}f$. Now using the Lemma 5 we obtain the thesis.
Theorem 4.

Let $f : M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine $\tilde{J}$-tangent vector field $C = -\frac{\partial f}{\partial z}$. If distribution $\mathcal{D}$ is involutive then for every $x \in M$ there exists a para-complex centro-affine immersion $g : V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ such that $f$ can be expressed in the neighborhood of $x$ in the form

$$f(x_1, \ldots, x_{2n}, z) = \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z. \quad (34)$$

Moreover, if $g : V \to \mathbb{R}^{2n+2}$ is a para-complex centro-affine immersion then $f$ given by the formula (34) is an affine hypersurface with a centro-affine $\tilde{J}$-tangent vector field and involutive distribution $\mathcal{D}$.
Proof. Let $(\varphi, \xi, \eta)$ be an induced almost paracontact structure on $M$ induced by $C$. The Frobenius theorem implies that for every $x \in M$ there exist an open neighborhood $U \subset M$ of $x$ and linearly independent vector fields $X_1, \ldots, X_{2n}, X_{2n+1} = \xi \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, 2n + 1$. For every $i = 1, \ldots, 2n$ we have $X_i = D_i + \alpha_i \xi$ where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus we have

$$0 = [X_i, \xi] = [D_i, \xi] - \xi(\alpha_i)\xi.$$ 

Now (9) and (12) imply that $[D_i, \xi]$ and $\xi(\alpha_i) = 0$. We also have

$$0 = [X_i, X_j] = [D_i, D_j] - D_j(\alpha_i)\xi + D_i(\alpha_j)\xi$$

for $i = 1, \ldots, 2n$.  

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Zuzanna Szancer

Centro-Affine hypersurfaces with an induced a. p. s.
Since $\mathcal{D}$ is involutive the above equality implies $[D_i, D_j] = 0$ for $i, j = 1, \ldots, 2n$. Of course the vector fields $D_1, \ldots, D_{2n}, \xi$ are linearly independent, so there exists a map $\psi(x_1, \ldots, x_{2n}, z)$ on $U$ such that
\[
\frac{\partial}{\partial z} = \xi, \quad \frac{\partial}{\partial x_i} = D_i, \quad i = 1, \ldots, 2n.
\]
Now applying the Lemma 5 we find that $f$ can be locally expressed in the form
\[
f(x_1, \ldots, x_{2n}, z) = \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z
\]
where $g : V \to \mathbb{R}^{2n+2}$ is an immersion defined on an open subset $V \subset \mathbb{R}^{2n}$. Moreover, since $\frac{\partial}{\partial x_i} \in \mathcal{D}$ we have that
\[
f_{x_i} = \tilde{J}g_{x_i} \cosh z - g_{x_i} \sinh z \in f_*(D).
\]
Since $f_*(D)$ is $\tilde{J}$ invariant we also have
\[ \tilde{J}f_{x_i} = g_{x_i} \cosh z - \tilde{J}g_{x_i} \sinh z \in f_*(D). \]

The above implies that $g_{x_i} \in f_*(D)$ for $i = 1, \ldots, 2n$. But, since $\{g_{x_i}\}$ are linearly independent they form basis of $f_*(D)$ (dim $f_*(D) = 2n$) so
\[ f_*(D) = \text{span}\{g_{x_1}, \ldots, g_{x_{2n}}\}. \]

Since $f_*(D)$ is $\tilde{J}$-invariant we also have that
\[ \tilde{J}g_{x_i} \in f_*(D) = \text{span}\{g_{x_1}, \ldots, g_{x_{2n}}\}. \]
that is $\tilde{J}g_{x_i} = \sum \alpha_i g_{x_i}$ where $\alpha_i \in C^\infty(U)$. Since $g$ do not depend on variable $z$ the functions $\alpha_i$ also do not, thus $\alpha_i \in C^\infty(V)$. In this way we have shown that for $g : V \to \mathbb{R}^{2n+2}$ the tangent space $TV$ is $\tilde{J}$-invariant (we can transfer $\tilde{J}$ from $g_*(TV)$ to $TV$). Since $\tilde{J}|_{f^*(D)}$ is para-complex and $f^*(D) = \text{span}_{C^\infty(U)}\{g_{x_1}, \ldots, g_{x_{2n}}\}$, so $\tilde{J}$ is para-complex on $TV$.

Finally $g$ is para holomorphic. Since $f$ is immersion $\{g_{x_1}, \ldots, g_{x_{2n}}, \tilde{J}g\}$ are linearly independent. Moreover, since $f$ is centro-affine we also have that $g$ is linearly independent with $\{g_{x_1}, \ldots, g_{x_{2n}}, \tilde{J}g\}$ that is $\{g, \tilde{J}g\}$ forms $\tilde{J}$-invariant transversal bundle to $g_*(TV)$. That is $g$ is a para-complex affine immersion.
In order to prove the second part of the theorem note that since $g$ is centro-affine para-complex affine immersion then $\{f_{x_1}, \ldots, f_{x_{2n}}, f_z, f\}$ are linearly independent. It means that $f$ is an immersion and is centro-affine. Moreover, $f$ is $\tilde{J}$-tangent since $\tilde{J}(\vec{of}) = -g \cosh z + \tilde{J}g \sinh z = f_z$. In particular $g$ is para holomorphic that is we have $\tilde{J}g_{x_i} = \sum_{j=1}^{2n} \alpha_{ij} g_{x_j}$ for $i = 1, \ldots, 2n$. Now by straightforward computations we get $\sum_{j=1}^{2n} \alpha_{ij} f_{x_j} = \tilde{J}f_{x_i}$ for $i = 1, \ldots, 2n$. That is $\tilde{J}f_{x_i} \in \text{span}\{f_{x_1}, \ldots, f_{x_{2n}}\}$. In this way we have shown that $\text{span}\{f_{x_1}, \ldots, f_{x_{2n}}\}$ is $\tilde{J}$-invariant and since its dimension is $2n$ it must be equal to $f_*(D)$. Now it is easy to see that $D = \{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}} \}$ is involutive as generated by canonical vector fields.
Theorem 5.

There are no improper $\tilde{J}$-tangent affine hyperspheres.

Proof. We have $\eta(SX) = -h(X, \xi)$ for all $X \in \mathcal{X}(M)$. Thus, if $S = 0$, $h(X, \xi) = 0$ for every $X \in \mathcal{X}(M)$, which contradicts nondegeneracy of $h$. 

Theorem 6.

Let $f : M \rightarrow \mathbb{R}^{2n+2}$ be a $\tilde{J}$-tangent affine hypersphere with an involutive distribution $\mathcal{D}$. Then $f$ can be locally expressed in the form:

$$f(x_1, \ldots, x_{2n}, z) = \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z$$

(35)

where $g$ is a proper para-complex affine hypersphere. Moreover, the converse is also true in the sense that if $g$ is a proper para-complex affine hypersphere then $f$ given by the formula (35) is a $\tilde{J}$-tangent affine hypersphere with an involutive distribution $\mathcal{D}$. 
Proof. (⇒) First note that due to Theorem 5 $f$ must be a proper affine hypersphere. Let $C$ be a $\tilde{J}$-tangent affine normal field. There exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $C = -\lambda f$. Since $C$ is $\tilde{J}$-tangent and transversal the same is $\frac{1}{\lambda} C = -f$. Thus $f$ satisfies assumptions of Theorem 4. By Theorem 4 there exists a para-complex centro-affine immersion $g: V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ and there exists an open interval $I$ such that $f$ can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z \quad (36)$$

for $(x_1, \ldots, x_{2n}) \in V$ and $z \in I$. 
Let $\zeta := -|\lambda|^{\frac{2n+3}{2n+4}} g$ and let $\nabla, h_1, h_2, S, \tau_1, \tau_2$ be induced objects on $V$ by $\zeta$. Using the Weingarten formula for $g$ and $\zeta$ we get

$$D_{\partial x_i} \zeta = -g_\ast (S \partial x_i) + \tau_1 (\partial x_i) \zeta + \tau_2 (\partial x_i) J \zeta.$$  

On the other hand, by straightforward computations we have

$$D_{\partial x_i} \zeta = \partial x_i (\zeta) = -|\lambda|^{\frac{2n+3}{2n+4}} g_\ast (\partial x_i).$$

Thus we obtain

$$S = |\lambda|^{\frac{2n+3}{2n+4}} I, \quad \tau_1 = 0, \quad \tau_2 = 0.$$ \hfill (37)
Now, to prove that $\zeta$ is an affine normal vector field it is enough to show that $|H_\zeta| = 1$. Since $g$ is para-holomorphic, without loss of generality, we may assume that

$$\partial_{x_{n+i}} = \tilde{J}\partial_{x_i}$$

for $i = 1 \ldots n$. Let

$$a := \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_n}, \tilde{J}\partial_{x_1}, \ldots, \tilde{J}\partial_{x_n}).$$

Then

$$\frac{1}{a} \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \tilde{J}\partial_{x_1}, \ldots, \tilde{J}\partial_{x_n}$$

is a unimodular basis relative to $\theta_\zeta$. 
Now

\[ H_\zeta = \frac{1}{a^2} \det \begin{bmatrix}
    h_1(\partial_{x_1}, \partial_{x_1}) & h_1(\partial_{x_1}, \partial_{x_2}) & \cdots & h_1(\partial_{x_1}, \partial_{x_{2n}}) \\
    h_1(\partial_{x_2}, \partial_{x_1}) & h_1(\partial_{x_2}, \partial_{x_2}) & \cdots & h_1(\partial_{x_2}, \partial_{x_{2n}}) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_1(\partial_{x_{2n}}, \partial_{x_1}) & h_1(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h_1(\partial_{x_{2n}}, \partial_{x_{2n}})
\end{bmatrix}. \]

By the Gauss formula for \( g \) we have

\[ g_{x_i x_j} = g^* (\nabla \partial_{x_i}, \partial_{x_j}) + h_1(\partial_{x_i}, \partial_{x_j}) \zeta \]

\[ + h_2(\partial_{x_i}, \partial_{x_j}) \tilde{J}_\zeta \quad \text{(38)} \]

\[ = g^* (\nabla \partial_{x_i}, \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j}) g - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i}, \partial_{x_j}) \tilde{J} g. \quad \text{(39)} \]
Let $\nabla$ and $\bar{h}$ be an induced connection and the second fundamental form for $f$. By the Gauss formula for $f$ we have

$$f_{x_ix_j} = \tilde{J}g_{x_ix_j} \cosh z - g_{x_ix_j} \sinh z$$

$$= f_*(\nabla_{\partial x_i} \partial x_j)$$

$$- \lambda \bar{h}(\partial x_i, \partial x_j)(\tilde{J}g \cosh z - g \sinh z).$$

(40)

(41)

Applying (38) to (40) we get

$$f_*(\nabla_{\partial x_i} \partial x_j) - \lambda \bar{h}(\partial x_i, \partial x_j)(\tilde{J}g \cosh z - g \sinh z)$$

$$= \tilde{J}g_*(\nabla_{\partial x_i} \partial x_j) \cosh z - g_*(\nabla_{\partial x_i} \partial x_j) \sinh z$$

$$- |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial x_i, \partial x_j)\tilde{J}g + h_2(\partial x_i, \partial x_j)g) \cosh z$$

$$+ |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial x_i, \partial x_j)g + h_2(\partial x_i, \partial x_j)\tilde{J}g) \sinh z$$

$$= f_*(\nabla_{\partial x_i} \partial x_j) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial x_i, \partial x_j)(\tilde{J}g \cosh z - g \sinh z)$$

$$- |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial x_i, \partial x_j)(g \cosh z - \tilde{J}g \sinh z)$$

$$= f_*(\nabla_{\partial x_i} \partial x_j) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial x_i, \partial x_j) \cdot f$$

$$- |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial x_i, \partial x_j) \cdot \tilde{J}f.$$
Since $f_*(\nabla_{\partial x_i} \partial x_j)$ as well as $\tilde{J}f$ are tangent we immediately obtain
\[-\lambda h(\partial x_i, \partial x_j) = -|\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial x_i, \partial x_j).\]
By the Gauss formula for $f$ we also have
\[h(\partial z, \partial z) = -\frac{1}{\lambda}\]
and
\[h(\partial z, \partial x_i) = h(\partial x_i, \partial z) = 0\]
for $i = 1 \ldots 2n$. 
Hence

\[
\det h := \begin{bmatrix}
    h(\partial_{x_1}, \partial_{x_1}) & h(\partial_{x_1}, \partial_{x_2}) & \cdots & h(\partial_{x_1}, \partial_{x_{2n}}) & 0 \\
    h(\partial_{x_2}, \partial_{x_1}) & h(\partial_{x_2}, \partial_{x_2}) & \cdots & h(\partial_{x_2}, \partial_{x_{2n}}) & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    h(\partial_{x_{2n}}, \partial_{x_1}) & h(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h(\partial_{x_{2n}}, \partial_{x_{2n}}) & 0 \\
    0 & 0 & \cdots & 0 & -\frac{1}{\lambda}
\end{bmatrix}
\]

\[
= -\frac{1}{\lambda} \det[h(\partial_{x_i}, \partial_{x_j})] = -\frac{1}{\lambda} \cdot \left(\frac{1}{\lambda} \cdot |\lambda|^{\frac{2n+3}{2n+4}}\right)^{2n} \det[h_1(\partial_{x_i}, \partial_{x_j})]
\]

\[
= -\frac{1}{\lambda} \cdot |\lambda|^{-\frac{2n}{2n+4}} a^2 H_\zeta.
\]
Now

\[(\omega_h)^2 = |\det h| = |\lambda|^{-\frac{2n-2}{n+2}} a^2 |H_\zeta| \]

(42)

On the other hand we have

\[
\omega_h = \theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \ldots, f_{x_{2n}}, f_z, C]
\]

\[
= -\lambda \det[\tilde{J}g_{x_1} \cosh z - g_{x_1} \sinh z, \ldots, \tilde{J}g_{x_{2n}} \cosh z - g_{x_{2n}} \sinh z, \tilde{J}g \sinh z - g \cosh z, \tilde{J}g \cosh z - g \sinh z].
\]

Since determinant is \((2n + 2)\)-linear and antisymmetric and since

\[
g_{x_{n+i}} = \tilde{J}g_{x_i} \text{ for } i = 1 \ldots n \text{ eventually we obtain}
\]

\[
\omega_h = -\lambda \det[g_{x_1}, \ldots, g_{x_n}, \tilde{J}g_{x_1}, \ldots, \tilde{J}g_{x_n}, g, \tilde{J}g]
\]

\[
= -\lambda (|\lambda|^{\frac{2n+3}{2n+4}})^{-2} \det[g_*(\partial_{x_1}), \ldots, g_*(\partial_{x_{2n}}), \zeta, \tilde{J}\zeta]
\]

\[
= -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}) = -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) a.
\]
Using the above formula in (42) we easily obtain

\[ |H_\zeta| = a^{-2} |\lambda|^{\frac{2n+2}{n+2}} \cdot \lambda^2 \cdot |\lambda|^{-\frac{4n+6}{n+2}} a^2 = 1. \]
Let \( g : U \to \mathbb{R}^{2n+2} \) be a proper para-complex affine hypersphere. Since \( g \) is a proper para-complex affine hypersphere there exists \( \alpha \neq 0 \) such that \( \zeta = -\alpha g \) is an affine normal vector field. Without loss of generality we may assume that \( \alpha > 0 \). Of course both \( g \) and \( \tilde{J}g \) are transversal thus \( \{g_{x_1}, \ldots, g_{x_{2n}}, g, \tilde{J}g\} \) form the basis of \( \mathbb{R}^{2n+2} \). The above implies that

\[
\begin{align*}
f : U \times I &\ni (x_1, \ldots, x_{2n}, z) \mapsto f(x_1, \ldots, x_{2n}, z) \in \mathbb{R}^{2n+2} \\
f(x_1, \ldots, x_{2n}, z) &:= \tilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z
\end{align*}
\]

is an immersion and \( C := -\alpha ^{\frac{2n+4}{2n+3}} \cdot f \) is a transversal vector field.
Field $C$ is $\tilde{J}$-tangent because $\tilde{J}C = \alpha^{\frac{2n+4}{2n+3}} f_z$. Since $C$ is equiaffine it is enough to show that $\omega_h = \theta$ for some positively oriented (relative to $\theta$) basis on $U \times I$. Let $\partial_{x_1}, \ldots, \partial_{x_2n}, \partial_z$ be a local coordinate system on $U \times I$. Since $g$ is para-holomorphic we may assume that $\partial_{x_{n+i}} = \tilde{J}\partial_{x_i}$ for $i = 1 \ldots n$.

Then we have

$$
\theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \ldots, f_{x_{2n}}, f_z, -\alpha^{\frac{2n+4}{2n+3}} f]$

$$
= -\alpha^{\frac{2n+4}{2n+3}} \det[\tilde{J}g_{x_1} \cosh z - g_{x_1} \sinh z, \ldots, \tilde{J}g_{x_{2n}} \cosh z - g_{x_{2n}} \sinh z,$$

$$
+ \tilde{J}g \sinh z - g \cosh z, \tilde{J}g \cosh z - g \sinh z]
$$

$$
= -\alpha^{\frac{2n+4}{2n+3}} \det[g_*(\partial_{x_1}), \ldots, g_*(\partial_{x_{2n}}), g, \tilde{J}g]
$$

$$
= -\alpha^{\frac{2n+4}{2n+3}} \cdot \frac{1}{\alpha^2} \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})
$$

$$
= -\alpha^{-\frac{2n+2}{2n+3}} \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}).
$$
In a similar way, like in the proof of the first implication we compute
\[
\det h = \alpha^{-\frac{2n+4}{2n+3}} \cdot \left( \frac{\alpha}{\alpha} \right)^{2n} \det h_1
\]
\[
= \alpha^{-\frac{2n+4}{2n+3}} \cdot \alpha^{-\frac{2n}{2n+3}} \det h_1
\]
\[
= \alpha^{-\frac{4n-4}{2n+3}} \det h_1.
\]
The above implies that
\[
(\omega_h)^2 = |\det h| = \alpha^{-\frac{4n-4}{2n+3}} |\det h_1|.
\]
Since
\[
|\det h_1| = |H_\zeta|[\theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})]^2
\]
we obtain
\[
(\omega_h)^2 = \alpha^{-\frac{4n-4}{2n+3}} |H_\zeta|[\theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})]^2.
\]
Finally, using the fact that $|H_\zeta| = 1$, we get $\omega_h = |\theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z)|$.
The proof is completed.
References


Thank you!