On Two Types of Slightly Countable Dense Homogeneous Spaces

by

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Part I

Definition: [1] A space (X, τ) **is** ***homogeneous*** **if for any two points** $x, y \in X$ **there exists a homeomorphism** $f : (X, \tau) \to (X, \tau)$ **such that** $f(x) = y$**.**

Part II

Definition: [2] A space (X, τ) is *countable dense homogeneous* (abbreviated: CDH) if it is separable space and for any two countable dense subsets A, B of X, there is a homeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(A) = B$.

Part III

Definition: [7] A function $f : (X, \tau_1) \to (Y, \tau_2)$ is *slightly continuous* if the inverse image of every clopen subset of (Y, τ_2) is a clopen subset of (X, τ_1).

Part IV

Definition: [8] A function $f : (X, \tau_1) \to (Y, \tau_2)$ is *slight homeomorphism* if f is a bijection and f and f^{-1} are slightly continuous.

Part V

Definition. [8] A space (X, τ) **is said to be** ***slightly homogeneous*** **if for any two points** $x, y \in X$**, there exists a slight homeomorphism** $f : (X, \tau) \to (X, \tau)$ **such that** $f(x) = y$**. A subset of a space** (X, τ)**, which has the form** $SC_x = \{y \in X :$ **there is a slight homeomorphism** $f : (X, \tau) \to (X, \tau)$ **such that** $f(x) = y\}$ **is called the slightly homogeneous component of** X **at** x**.**

Part VI

Definition. [8] A separable space (X,τ) is said to be *slightly countable dense homogeneous* (abbreviated: SCDH) if given any two countable dense subsets A, B of X, there is a slight homeomorphism $f:(X,\tau)\rightarrow(X,\tau)$ such that $f(A)=B$.

Part VII

Theorem [8] Let U be a non-empty clopen subset of a space (X, τ). If SC_x is a slightly homogeneous component of $x \in X$ and $U \subseteq SC_x$, then SC_x is open in X.

Theorem (a) [8] Every CDH space is SCDH but not conversely.

(b) [4] Every zero dimensional SCDH space is CDH.

Part VIII

Definition Let (X, τ) be a space. A subset $A \subseteq X$ is said to be *slightly dense* if for every non-empty clopen set $U \subseteq X$, $U \cap A \neq \emptyset$.

Part IX

Remark. Dense subsets of a space (X, τ) are slightly dense.

Part X

Example. Consider the space $((0,1)\cup(2,3),\tau_u)$ and $D=\{\frac{1}{2},\frac{5}{2}\}$. D is a slightly dense set that is not dense.

Part XI

Theorem. Let (X, τ) be a zero dimensional space and let $D \subseteq X$. Then D is dense in X iff it is slightly dense in X.

Part XII

Theorem. **Let (X, τ) be a space such that for some $x \in X$, SC_x is not open. Then $X - SC_x$ is slightly dense in (X, τ).**

Part XIII

Definition. For every finite non zero cardinal number n, denote the set $\{A \subseteq X : A$ is slightly dense and $|A| = n\}$ by C_n, and denote the set $\{A \subseteq X : A$ is slightly dense and $|A| = \aleph_0\}$ by C_∞.

Theorem. Let (X, τ) be a space. Then the following are equivalent.

(i) (X, τ) is connected.

(ii) A is slightly dense for all non-empty set $A \subseteq X$.

(iii) $\{x\}$ is slightly dense for all $x \in X$.

(iv) $C_1 \neq \emptyset$.

Part XIV

Theorem. Let (X, τ) be a disconnected space. If for some non-zero finite cardinal number n, $C_n = \{A \subseteq X : |A| = n\}$, then $|X| \leq 2n - 2$.

Part XV

Corollary. If X is a set with $|X| > 2$ and τ is a topology on X, then (X, τ) is connected iff $C_2 = \{A \subseteq X : |A| = 2\}$.

Part XVI

Corollary. Let (X, τ) be a space such that $|X| > 2n - 2$ where n is a non-zero finite cardinal number. If $C_n = \{A \subseteq X : |A| = n\}$, then (X, τ) is connected.

Part XVII

Definition. A space (X, τ) is said to be slightly separable if it contains a countable slightly dense subset.

Remarks. 1. A space (X, τ) is slightly separable iff $C_n \neq \emptyset$ for some $n \in \mathbb{N} \cup \{\infty\}$.

2. Every connected space is slightly separable but not conversely.

3. Every separable space is slightly separable but not conversely.

4. A zero dimensional space is separable iff it is slightly separable space.

5. The slightly continuous image of a slightly separable space is slightly separable.

Part XVIII

Theorem. A clopen subspace of a slightly separable space is slightly separable.

Part XIX

Definition. [12] Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a collection of spaces such that $X_\alpha \cap X_\beta = \emptyset$ for all $\alpha \neq \beta$. Let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$ be topologized by $\{U \subseteq X : U \cap X_\alpha \in \tau_\alpha$ for all $\alpha \in \Lambda\}$. Then (X, τ) is called the disjoint sum of the spaces $(X_\alpha, \tau_\alpha), \alpha \in \Lambda$.

Part XX

Theorem. Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ **be a family of spaces with** $X_\gamma \cap X_\beta = \emptyset$ **for** $\gamma \neq \beta$**. If for all** $\alpha \in \Lambda$**,** (X_α, τ_α) **contains a non-empty slightly dense set** D_α**, then** $\bigcup_{\alpha\in\Lambda} D_\alpha$ **is slightly dense in the disjoint sum space** $(\bigcup_{\alpha\in\Lambda} X_\alpha, \tau_d)$**.**

Part XXI

Definition. **A space (X, τ) is said to be *slightly countable dense homogeneous* of type (2) (SCDH(2)) if (X, τ) is slightly separable and for any two countable slightly dense sets A and B in X, there exists a slight homeomorphism $h : (X, \tau) \rightarrow (X, \tau)$ such that $h(A) = B$.**

Part XXII

Theorem. If (X, τ) is SCDH(2), then every slightly dense subset of X different from X is infinite.

Part XXIII

Corollary. Let (X, τ) be a connected space. Then (X, τ) is SCDH(2) iff $|X| = 1$.

Part XXIV

Example. $(\mathbb{R}, \tau_u)$ is a CDH space that is not SCDH(2).

Part XXV

Theorem. A zero dimensional space is CDH iff it is SCDH(2).

Part XXVI

Example. The space $(\mathbb{Q}^c, \tau_u)$ is SCDH(2).

Part XXVII

Theorem. Let (X,τ) be a SCDH(2) space. Then X is countable iff

$$\tau = \tau_{disc}.$$

Part XXVIII

Example. The space $(\mathbb{Q}, \tau_u)$ is not SCDH(2).

Part XXIX

Theorem. A zero dimensional space (X, τ) is SCDH(2) iff it is SCDH.

Part XXX

Theorem. Let $\{(X_n, \tau_n) : n \in \mathbb{N}\}$ **be a family of SCDH(2) spaces with** $X_n \cap X_m = \emptyset$ **for** $n \neq m$**. Then the disjoint sum space** $(\bigcup_{n=1}^{\infty} X_n, \tau_d)$ **is SCDH(2).**

Part XXXI

Definition. A space (X, τ) **is said to be** ***slightly countable dense homogeneous of type (3)*** **(SCDH(3)) if** (X, τ) **is slightly separable and for each** $n \in \mathbb{N} \cup \{\infty\}$ **and** $A, B \in C_n$**, there exists a slight homeomorphism** $f : (X, \tau) \to (X, \tau)$ **such that** $f(A) = B$**.**

Part XXXII

Theorem. Every connected space is SCDH(3).

Part XXXIII

Theorem. Every SCDH(2) space is SCDH(3).

Part XXXIV

Example. $(\mathbb{R}, \tau_u)$ is SCDH(3) but not SCDH(2).

Part XXXV

Theorem. Every zero dimensional CDH space is SCDH(3).

Part XXXVI

Theorem. Let (X, τ) be a space such that all slightly dense sets in X have the same cardinality, then (X, τ) is SCDH(2) iff it is SCDH(3).

References

[1] W. Sierpenski, *Sur uni proprit'te' topologique des ensembles de'nombrable dense en soi*. Fund. Math **1** (1920), 11-28.

[2] R. Bennett, *Countable dense homogeneous spaces.* Fund. Math. **74** (1972), 189–194.

[3] A. V. Arhangel'skii and J. van Mill, *On the cardinality of countable dense homogeneous spaces.* Proc. Amer. Math. Soc. **141** (2013), 4031–4038.

[4] R. Hernandez-Gutiérrez and M. Hrušak, *Non-meager P-filters are countable dense homogeneous.* Colloq. Math. **130** (2013), 281–289.

[5] M. Hrusak and J. van Mill, *Nearly countable dense homogeneous spaces.* Canad. J. Math. **66** (2014), 743–758.

[6] D. Repovš, L. Zdomskyy and S. Zhang, *Countable dense homogeneous filters and the Menger covering property.* Fund. Math. **224** (2014), 233–240.

[7] A. R. Singal and R. C. Jain, *Slightly continuous mappings.* J. Indian Math. Soc. (N.S.) **64** (1997), 195–203.

[8] S. Al Ghour and N. Al Khatib, *On slight homogeneous and countable dense homogeneous spaces.* Mat. Vesnik **63** (2011), 133–144.

[9] S. Al Ghour, *Minimality and prehomogeneity.* Acta Math. Univ. Comenian **72** (2003), 237-244.

[10] S. Al Ghour, *Components and local prehomogeneity.* Acta Math. Univ. Comenian. (N.S.) **73** (2004), 187–196.

[11] S. Al Ghour, K. Zoubi and A. Fora, *Semihomogeneous topological spaces.* Publ. Math. Debrecen **67** (2005), 131-142.

[12] R. Engelking, *General Topology*, Hilderman-Berlin, 1989.

[13] M. Yassien, *Homogeneous spaces.* Master Thesis, Department of Mathematics, Yarmouk University, Irbid-Jordan, 1995.