Introduction

- Einstein-Weyl causality, that is, relativistic causality, has consequences for the topology of space-time. This was studied by Borchers and Sen. In the 1990's they rigorously investigated the mathematical implications of causality and proved that a denumerable space-time would be admitted. They showed subsequently that the notion of causality could effectively be extended to discontinuua but they were left with important questions regarding the nature of the physical line as opposed to the mathematical real line. In this talk, we return to their initial result and ask whether space-time could, in fact, be denumerable but also continuous. We find, if so, there are fundamental implications for quantum mechanics.

The Implications of Einstein-Weyl Causality for Quantum Mechanics

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To fulfill the topological properties of space-time required by Einstein-Weyl causality, we employ a constructible foundation.

ZF – Axiom Schema of Subsets – Power Set + ω*-Constructibility

Extensionality	Two sets with just the same members are equal.
Pairs	For every two sets, there is a set that contains just them.
Union	For every set of sets, there is a set with just all their members.
Infinity	There is at least one set ω* which contains the null set and for every member there is a next member that contains just all its predecessors
Regularity	Every non-empty set has a minimal member (i.e. “weak” regularity).
Replacement	Replacing members of a set one-for-one creates a set. (i.e. bijective replacement)
ω*-Constructibility	The subsets of ω* are constructible

The ω*-constructibility axiom well-orders the set of constructible subsets of ω*, giving a metric space R*. Its members mirror "binimals" (e.g., .0110011…).

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Q* is an enlargement of the usual set of rational numbers Q.

***Def*: Two members of Q* are "identical" if their ratio is 1. We can now employ the symbol "≡" for "is identical to".**

***Def*: Letting y be the member Q* and employing the symbol "=" to signify "equals", $y = 0 \leftrightarrow \forall n[y < 1/n]$ where n is a finite natural number. Note: $x \equiv y \rightarrow x = y$ but not the converse.**

An *equality-preserving* bijective map $\Phi(x,u)$ between intervals X and U of **R*** in which $x \in X$ and $u \in U$

$$\forall x_1, x_2, u_1, u_2 \left[\Phi(x_1,u_1) \wedge \Phi(x_2,u_2) \rightarrow [x_1 - x_2 = 0 \leftrightarrow u_1 - u_2 = 0] \right]$$

creates pieces biunique and homeomorphic with **R***.

The range vanishes if and only if the domain vanishes.

- **$u(x)$ is a function of $x \in$ R* if and only if it is either a constant or a finite number of connected biunique pieces such that some m^{th} derivative is a constant. Thus, $u(x)$ is a polynomial and each of its derivatives would be obtained by using $\frac{dx^n}{dx} = nx^{n-1}$ term-by-term; it is uniformly continuous and locally homeomorphic with R*. All $u(x)$ have the important property**

$\neg$range $u(x) \equiv 0 \rightarrow$ [range $u(x) \neq 0 \leftrightarrow$ domain $u(x) \neq 0$]

Uniformly continuous functions that have no finite derivative that is constant can still be approached as closely as necessary by polynomials of sufficiently high degree obtained by iteratively minimizing λ using the following algorithm in which integration is defined as the inverse of differentiation:

$$\int_a^b [p\left(\frac{du}{dx}\right)^2 - qu^2]dx \equiv \lambda; \quad \int_a^b ru^2 dx \equiv Const.$$

$$a \neq b;\ u\frac{du}{dx} \equiv 0 \text{ at a and b; p, q and r are polynomials in x.}$$

The polynomials so obtained by this algorithm are effectively Sturm-Liouville eigenfunctions. They are of bounded variation and locally homeomorphic with R*. The amount of iterations can be arbitrarily large but finite.

This has both physical and philosophical implications.

Consider a string of finite length:

$$\int \left[\left(\frac{\partial u_x u_t}{\partial x} \right)^2 - \left(\frac{\partial u_x u_t}{\partial t} \right)^2 \right] dx\, dt \equiv 0$$

The u_x and u_t can be obtained subject to boundary conditions and to the constraint

$$\lambda_x - \lambda_t \equiv 0$$

This can be generalized to more complex fields and to finitely many dimensions of a compactified space-time.

Let $u_{\ell mi}(x_i)$ and $u_{\ell mj}(x_j)$ be eigenfunctions with positive eigenvalues $\lambda_{\ell mi}$ and $\lambda_{\ell mj}$ respectively.

We now define a “field” as a sum of eigenstates

$$\underline{\Psi}_m = \sum_{\ell} \Psi_{\ell m}\underline{i_{\ell}}, \Psi_{\ell m} = \prod_{\mathrm{i}} \mathrm{u}_{\ell mi} \prod_{\mathrm{j}} \mathrm{u}_{\ell mj}$$

with the novel physical postulate that the Lagrange integral over a ***compactified*** space-time is ***identical*** to 0 for every eigenstate**.** Let

$$ds d\tau = \prod_i r_i dx_i \prod_j r_j dx_j$$

$$\sum_{\square}\int \left\{\sum_i \frac{1}{r_i}\left[P_{\square mi}\left(\frac{\partial \Psi_{\square m}}{\partial x_i}\right)^2 - Q_{\square mi}\Psi_{\square m}^2\right] - \sum_j \frac{1}{r_j}\left[P_{\square mj}\left(\frac{\partial \Psi_{\square m}}{\partial x_j}\right)^2 - Q_{\square mj}\Psi_{\square m}^2\right]\right\} ds d\tau \equiv 0$$

.

Let both $$\sum_{m}\sum_{\ell}\int\left\{\sum_{i}\frac{1}{r_i}\left[P_{\ell mi}\left(\frac{\partial \Psi_{\ell m}}{\partial x_i}\right)^2 - Q_{\ell mi}\Psi_{\ell m}^2\right]\right\}ds\,d\tau$$ and

$$\sum_{m}\sum_{\ell}\int\left\{\sum_{j}\frac{1}{r_j}\left[P_{\ell mj}\left(\frac{\partial \Psi_{\ell m}}{\partial x_j}\right)^2 - Q_{\ell mj}\Psi_{\ell m}^2\right]\right\}ds\,d\tau$$ be represented by α(Ψ)

Assume P_{lmj}, $Q_{lmj} \geq 0$ and that domainΨ ≠ 0. α(Ψ) is non-negative and closed to addition.

I. Recall that if ¬rangeΨ ≡ 0, then rangeΨ ≠ 0 ↔ domainΨ ≠ 0.

II. rangeΨ ≡ 0 → α(Ψ) ≡ 0;¬rangeΨ ≡ 0 → rangeΨ ≠ 0 → α(Ψ) ≠ 0.
Thus α(Ψ) ≡ 0 or α(Ψ) ≠ 0; ∴ α(Ψ) is not continuous.

III. α(Ψ) has only discrete values: α(Ψ) ≡ $n\kappa$, where n is any integer and κ is some finite unit which must be determined empirically.

Moreover, κ has indeed been determined empirically (next slide).

Let $\ell = 1, 2$, $r_\ell = P_{1m\ell} = P_{2m\ell} = 1$, $Q_{1m\ell} = Q_{2m\ell} = 0$, $\tau = \omega_m t$ and we normalize Ψ as follows:

$$\Psi_m = \sqrt{(C/2\pi)} \prod_i u_{im}(x_i)[u_{1m}(\tau) + i \cdot u_{2m}(\tau)] \quad (9)$$

where $i = \sqrt{-1}$ with

$$\int \sum_m \prod_i u_{im}^2 ds (u_{1m}^2 + u_{2m}^2) \equiv 1 \qquad (10)$$

then:

$$\frac{du_{1m}}{d\tau} = -u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = u_{1m} \qquad (11)$$

or

$$\frac{du_{1m}}{d\tau} = u_{2m} \quad \text{and} \quad \frac{du_{2m}}{d\tau} = -u_{1m} \qquad (12)$$

For the minimal non-vanishing field, α is just the least finite value κ. Thus,

$$(C/2\pi) \sum_m \oint \int \left[\left(\frac{du_{1m}}{d\tau} \right)^2 + \left(\frac{du_{2m}}{d\tau} \right)^2 \right]$$
$$\prod_i u_{im}^2(x_i) ds d\tau \equiv C \equiv \kappa \qquad (13)$$

Substituting the Planck constant h for κ, this can now be put into the conventional Lagrangian form for the time term in the Schrödinger equation,

$$\frac{h}{2i} \sum_m \oint \int \left[\Psi_m^* \left(\frac{\partial \Psi_m}{\partial t} \right) - \left(\frac{\partial \Psi_m^*}{\partial t} \right) \Psi_m \right] ds dt$$

Assume $\exists\Psi\ \neg\text{range}\Psi \equiv 0$ and domainΨ is all of space-time. With this we have: all of space-time $\neq 0 \rightarrow \exists\Psi\ \text{domain}\Psi \neq 0$.

Since domain$\Psi \neq 0 \leftrightarrow$ range$\Psi \neq 0$, we get $\exists\Psi\ \text{range}\Psi \neq 0 \rightarrow \exists\Psi\ \alpha(\Psi) \neq 0$ as we got before, so that : all of space-time $\neq 0 \rightarrow \exists\Psi\ \alpha(\Psi) \neq 0$

Also, if all of space-time = 0, the upper and lower limits of all the integrals in the computation of $\alpha(\Psi)$ are equal so that:

all of space-time = 0 $\rightarrow \neg\exists\Psi\ \alpha(\Psi) \neq 0$.

Therefore: all of space-time $\neq 0 \leftrightarrow \exists\Psi\ \alpha(\Psi) \neq 0$.

This is a definition of a relational space-time.

Furthermore, since we have shown before that $\alpha(\Psi) \equiv nh$, it follows that: all of space-time $\neq 0 \leftrightarrow \exists\Psi\ \alpha(\Psi) \geq h$.

$\alpha(\Psi) \geq h$ is the Uncertainty Principle.

Dyson (using ZF) argued that if the QED perturbation series converges to a limit it will create a catastrophically unstable vacuum state; thus the series must be divergent. However, this theory has no induction theorem thus no mathematical limit can be reached and no unstable vacuum state will be created. This is a seminal example of this "new physics" removing infinities.

Finally, it had been shown by Börchers/Sen that Q^2 is an ordered space that fulfills the strict Hausdorff topology requirements for Einstein-Weyl causality.

In this theory, by definition Q^2 is embedded in R^{*2} and we have shown that all functions are locally homeomorphic with R*, thus R^{*2}, or more generally R^{*n}, exactly fulfills all the requirements for Einstein-Weyl causality in a space-time that is both denumerable and also continuous.

Principal Conclusions

In this theory, space-time is constructible and its topological properties fulfill the requirements for Einstein-Weyl causality. When field functions are included in the theory, the Schrödinger equation emerges. Thus quantum mechanics itself provides confirmation for the constructibility of space-time and the avoidance of infinities in this theory may give a "new physics" alternative to renormalization.

For Further Discussion

- These results together infer that the Schrödinger equation is conceptually cumulative with prior physics.
- By relating Einstein-Weyl causality to the Schrödinger equation, we may have discovered a useful bridge between relativity and quantum theory.
- Regarding Wigner's meta-physical question on the unreasonable effectiveness of mathematics in physics: We have shown that the Schrödinger equation, relativistic causality, a relational space-time and regularization jointly emerge from a bottom-up mathematical approach.

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ZF – Axiom Schema of Subsets – Power Set + ω^*-Constructibility

Extensionality- Two sets with just the same members are equal.

$$\forall x \forall y [\forall z [z \in x \leftrightarrow z \in y] \rightarrow x = y]$$

Pairs- For every two sets, there is a set that contains just them.

$$\forall x \forall y \exists z [\forall w w \in z \leftrightarrow w = x \vee w = y]$$

Union- For every set of sets, there is a set with just all their members.

$$\forall x \exists y \forall z [z \in y \leftrightarrow \exists u [z \in u \wedge u \in x]]$$

Infinity- There is at least one set ω^*which contains the null set and for every member there is a next member that contains just all its predecessors

$$\exists \omega * [0 \in \omega * \wedge \forall x [x \in \omega * \rightarrow x \cup \{x\} \in \omega *]]$$

Regularity- Every non-empty set has a minimal member (i.e. "weak" regularity).

$$\forall x \left[\exists y y \in x \rightarrow \exists y [y \in x \wedge \forall z \neg [z \in x \wedge z \in y]]\right]$$

Replacement- Replacing members of a set one-for-one creates a set (i.e., "bijective" replacement).
Let f(x,y) a formula in which x and y are free,

$$\forall z \forall x \in z \exists y [\varphi(x,y) \wedge \forall u \in z \forall v [\varphi(u,v) \rightarrow u = x \leftrightarrow y = v]] \rightarrow \exists r \forall t [t \in r \leftrightarrow \exists s \in z \varphi(s,t)]$$

ω^*-Constructibility- The subsets of ω^* are constructible

$$\forall \omega^* \exists S[(\omega^*,0) \in S \wedge \forall y{\neq}0 \forall z[(y,z) \in S \leftrightarrow ([y - m_y] \cup m_y , z \cup \{z\}) \in S]]$$

where m_y is the minimal member of y.

T = ZF – Power Set – Axiom Schema of Subsets + ω^*-Constructibility is consistent relative to ZFC^+

ZFC^+ = ZF + "All sets are Constructible" has been shown to be consistent by Goedel. We can obtain T as follows. We remove from ZF the Axiom Schema of Subsets (i.e., from the Axioms of Replacement and Regularity) and the Power Set axiom. In ZF, ω^* exists because of the Axiom of Infinity. If we can show that the subsets of $\omega^*$are constructed exactly as required by Goedel's axiom, the statement "The subsets of ω^* are constructible", as below, will not lead to a contradiction and thus can be adjoined as an axiom. Indeed, the subsets are generated sequentially and hence are well-ordered so that, in any set S of at least one of those subsets, there is a member obtained before any other member of S; this is Goedel's condition for the existence of constructible sets in ZFC^+. Thus we can delete the axiom "All sets are Constructible", leaving a sub-theory T consistent relative to ZFC^+ since any inconsistency in T would lead to an inconsistency in ZFC^+.
Note that we have also shown that there are denumerably many constructible subsets of ω^* in T.

ω^*-Constructibility- The subsets of ω^* are constructible

$$\forall\omega^*\exists S[(\omega^*,0) \in S \wedge \forall y\neq 0\forall z[(y,z) \in S \leftrightarrow ([y - m_y]\cup m_y, z\cup\{z\}) \in S]]$$

where m_y is the minimal member of y.

Generalized ω*-constructibility:

- $\forall \omega^* \exists S_n [(Z_{n-1}^*, 0) \in S_n \wedge \forall y_n \neq 0 \forall z_n [(y_n, z_n) \in S_n \leftrightarrow ((y_n - m_y) \cup m_y, z_n \cup \{z_n\}) \in S_n]]$, where m_y is the minimal element of y_n , $n \geq 1$ and Z_0^* is ω*

Z_n^* is given by $(0, Z_n^*) \in S_n$ and is equinumerous with ω*

$\mathbf{C_n}$ is created by the axiom of replacement from Z_n^* and the one-to-one mapping S_n . $\mathbf{C_0}$ is ω*. $\mathbf{C_1}$ is R*.

The constructible sets are $\bigcup_{\mathbf{N}} \mathbf{C_n}$ $\mathbf{N} \geq 0$

There are no impredicative sets

The theory is uniformly dependent on ω*

Representation of arithmetic tables in R*

O is represented by ω^*; Define y' by $(y - m_y) \cup m_y$

[x,y] and [x,y],z] represent ordered pairs.

addition: $x+y=z$

$\exists A \forall x \in R^* \forall y \in R^* E!z \in R^*([[O,O],O] \in A \wedge [[x,y],z] \in A \rightarrow [[x,y'],z'] \in A \wedge [[x',y],z'] \in A;$

multiplication: $x \cdot y = z$

$\exists M \forall x \in R^* \forall y \in R^* E!z \in R^*([[O,O],O] \in M \wedge [[x,y],z] \in M \rightarrow [[x,y'],z+x] \in M \wedge [[x',y],z+y] \in M);$

Representation of Q*

Define $[a,b]_r$ such that $[a_1,b]_r + [a_2,b]_r \equiv [a_1+a_2,b]_r$ and $[a_1,b_1]_r \equiv [a_2,b_2]_r \leftrightarrow a_1 \cdot b_2 \equiv a_2 \cdot b_1$.

The extended set of rationals Q* is the set of such pairs for all a and b in ω^*

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