Projective Bivector Parametrization of Isometries
Part II: Hamilton and Cayley's Contribution

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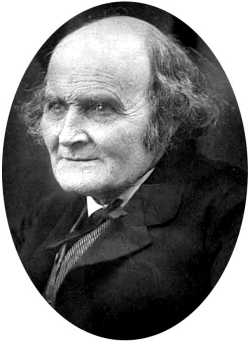
New Hope...

O. Rodrigues [1840]

W. Hamilton [1843]



A. Cayley [1846]

The Vector-Parameter

Some major advantages of Rodrigues' construction:

- compact expressions and no excessive parameters whatsoever
- topologically correct parametrization of $\mathrm{SO}(3) \cong \mathbb{RP}^3$, instead of coordinates on $\mathbb{T}^3$ (e.g., Euler angles), which yield singularities
- allows for rational expressions for the rotation's matrix entries

$$\mathcal{R}(\mathbf{c}) = \frac{(1 - \mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1 + \mathbf{c}^2}$$

- an efficient composition to replace the usual matrix multiplication

$$\langle\, \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)} \quad \Leftrightarrow \quad \mathcal{R}(\mathbf{c}_2)\,\mathcal{R}(\mathbf{c}_1) = \mathcal{R}(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle)$$

- numerically fast and analytically convenient representation.

Quaternions and the Spin Cover $\mathrm{SU}(2) \xrightarrow{\mathbb{Z}_2} \mathrm{SO}(3)$

We identify vectors $\mathbf{x} \in \mathbb{R}^3$ with imaginary (skew-hermitian) quaternions

$$\mathbf{x} \quad \longrightarrow \quad \mathbf{X} = x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k} \in \mathbb{H}.$$

Similarly, elements of $\mathrm{SU}(2) \cong \mathbb{S}^3$ are presented as unit quaternions

$$\mathbb{S}^3 \ni \zeta = (\zeta_\circ, \boldsymbol{\zeta}) = \zeta_\circ + \zeta_1\,\mathbf{i} + \zeta_2\,\mathbf{j} + \zeta_3\,\mathbf{k}, \qquad |\,\zeta\,|^2 = \det(\zeta) = 1.$$

Then, the adjoint action of $\mathbb{S}^3$ in its algebra $\mathbb{R}^3$

$$\mathrm{Ad}_\zeta : \mathbf{X} \longrightarrow \zeta\,\mathbf{X}\,\zeta^{-1}, \qquad \zeta^{-1} = \bar{\zeta} = (\zeta_\circ, -\zeta)$$

preserves metric and orientation, so it represents $\mathrm{SO}(3) \cong \mathbb{RP}^3$, namely as

$$\mathcal{R}(\zeta) = (\zeta_\circ^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta}\boldsymbol{\zeta}^t + 2\zeta_\circ\boldsymbol{\zeta}^\times.$$

The, the famous *Rodrigues'* rotation formula follows with the substitution

$$\zeta_\circ = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\,\mathbf{n}.$$

The Projective Map

Projecting onto the hyperplane $\zeta_\circ = 1$ we obtain the vector-parameter

$$\mathbf{c} = \frac{\boldsymbol{\zeta}}{\zeta_\circ} = \tau\mathbf{n} \in \mathbb{RP}^3, \qquad \tau = \tan\frac{\varphi}{2}$$

also known as *Rodrigues'* vector. Quaternion multiplication

$$(\xi_\circ, \boldsymbol{\xi}) \otimes (\zeta_\circ, \boldsymbol{\zeta}) \;\to\; (\xi_\circ\zeta_\circ - (\boldsymbol{\xi}, \boldsymbol{\zeta}),\, \xi_\circ\boldsymbol{\zeta} + \zeta_\circ\boldsymbol{\xi} + \boldsymbol{\xi} \times \boldsymbol{\zeta})$$

yields upon the above projection the efficient composition law

$$\langle\, \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)}.$$

that obviously constitutes a representation as it is associative and satisfies

$$\langle\, \mathbf{c},\, 0 \,\rangle = \langle\, 0,\, \mathbf{c} \,\rangle = \mathbf{c}, \qquad\qquad \langle\, \mathbf{c},\, -\mathbf{c} \,\rangle = 0.$$

Cayley's Transform

Instead of the exponential map one may use *Cayley's transform*

$$\mathrm{Cay}(\xi) = \frac{1+\xi}{1-\xi}$$

that maps the imaginary axis to the unit circle in $\mathbb{C}$. More generally, if ξ is skew-hemritian, $\mathrm{Cay}(\xi)$ is obviously unitary and

$$\mathrm{Cay}: \quad \mathfrak{so}(p,q) \longrightarrow \mathsf{SO}(p,q).$$

In the case of SO(3) we have

$$\mathrm{Cay}(\mathbf{c}^{\times}) = \exp(\mathbf{s}^{\times})$$

which reduces to a polynomial due to Hamilton-Cayley's theorem.

Lorentzian $2+1$ Space

We use the duality between the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$

- quaternions $(\mathbb{H}) \longrightarrow$ split quaternions $(\mathbb{H}')$
- Euclidean metric $\longrightarrow$ Lorentz metric

$$\mathbf{u}\cdot\mathbf{v} = (\mathbf{u}, \eta\,\mathbf{v}), \qquad \eta = \mathrm{diag}(1,1,-1)$$

$$\mathbf{c}^{\times} \to \mathbf{c}^{\curlywedge} = \eta\,\mathbf{c}^{\times} \in \mathfrak{so}(2,1), \qquad \mathbf{u}^{t} \to \mathbf{u}^{T} = \eta\,\mathbf{u}^{t}.$$

- the hyperbolic composition law

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2 \cdot \mathbf{c}_1}$$

rational expression for the pseudo-rotation matrix

$$\Lambda(\mathbf{c}) = \mathrm{Cay}(\mathbf{c}^{\curlywedge}) = \frac{(1+\mathbf{c}^2)\mathcal{I} - 2\,\mathbf{c}\mathbf{c}^{T} + 2\,\mathbf{c}^{\curlywedge}}{1-\mathbf{c}^2}.$$

Analogues of *Rodrigues'* Rotation Formula

Depending on the geometric type of the invariant axis $\Lambda(\mathbf{c})$ is

- *Hyperbolic:* $\mathrm{Tr}\,\Lambda > 3,\ \zeta^2 = \zeta_\circ^2 - 1 > 0$ (*space-like*) $\Rightarrow\ \tau = \mathrm{th}\,\dfrac{\varphi}{2}$

$$\Lambda(\mathbf{n},\varphi) = \mathrm{ch}\,\varphi\,\mathcal{I} + (1 - \mathrm{ch}\,\varphi)\,\mathbf{nn}^T + \mathrm{sh}\,\varphi\,\mathbf{n}^{\curlywedge}.$$

- *Elliptic:* $\mathrm{Tr}\,\Lambda < 3,\ \ \zeta^2 < 0$ (*time-like*) $\Rightarrow\ \tau = \tan\dfrac{\varphi}{2}$

$$\Lambda(\mathbf{n},\varphi) = \cos\varphi\,\mathcal{I} + (\cos\varphi - 1)\,\mathbf{nn}^T + \sin\varphi\,\mathbf{n}^{\curlywedge}.$$

- *Parabolic:* $\mathrm{Tr}\,\Lambda = 3,\ \ \zeta^2 = 0$ (*isotropic*) $\Rightarrow\ \tau = \dfrac{\varphi}{2}$

$$\Lambda(\mathbf{n},\varphi) = \mathcal{I} + \varphi\,\mathbf{n}^{\curlywedge} - \frac{\varphi^2}{2}\,\mathbf{nn}^T.$$

- *Non-Orthochronous:* $\Lambda_{33} < 0,\ \ \zeta^2 = \zeta_\circ^2 + 1\ \Rightarrow\ \tau = \coth\dfrac{\varphi}{2}$

$$\Lambda(\mathbf{n},\varphi) = -\mathrm{ch}\,\varphi\,\mathcal{I} + (1 + \mathrm{ch}\,\varphi)\,\mathbf{nn}^T - \mathrm{sh}\,\varphi\,\mathbf{n}^{\curlywedge}.$$

The Decomposition Problem

Adopting the notation $\epsilon_k = \hat{\mathbf{c}}_k^2$ we obtain the condition

$$\Delta = -\begin{vmatrix} \epsilon_1 & g_{12} & r_{31} \\ g_{21} & \epsilon_2 & g_{23} \\ r_{31} & g_{32} & \epsilon_3 \end{vmatrix} \geq 0$$

and the corresponding solutions in the form

$$\tau_k^{\pm} = \frac{\rho^k}{\omega_k \mp \sqrt{\Delta}}, \qquad \rho^k = \varepsilon^{ijk}(g_{ij} - r_{ij}), \quad i > j$$

with

$$\omega_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \Lambda^{-1}(\mathbf{c})\,\hat{\mathbf{c}}_3\right), \qquad \omega_2 = \omega, \qquad \omega_3 = \left(\Lambda(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\right)$$

and in the case of two axes

$$\tau_1 = \frac{r_{22} - \epsilon_2}{\mathring{\omega}_1}, \qquad \tau_2 = \frac{r_{11} - \epsilon_1}{\mathring{\omega}_2}$$

where we denote

$$\mathring{\omega}_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \Lambda^{-1}(\mathbf{c})\,\hat{\mathbf{c}}_2\right), \qquad \mathring{\omega}_2 = \left(\Lambda(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2\right).$$

Discriminant Conditions and Geometric Restrictions

The condition $\Delta \geq 0$ is necessary and sufficient only in the regular case. On the other hand, there is the gimbal lock singularity

$$\hat{\mathbf{c}}_3 = \pm\Lambda(\mathbf{c})\,\hat{\mathbf{c}}_1$$

in which the solutions are given by

$$\tau_2 = \frac{r_{11} - \epsilon_1}{\mathring{\omega}_2}\,, \qquad \tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1 \tau_1 \tau_3} = \frac{r_{22} - \epsilon_2}{\mathring{\omega}_1}$$

and it is not sufficient as $\Delta = \epsilon_1 (r_{21} - g_{21})^2 \geq 0$ does not imply the two-axes condition $r_{21} = g_{21}$ in the space-like and null cases $\epsilon_1 \geq 0$. We have the restrictions $|\tau_k| \neq 1$ in the space-like case $\epsilon_k = 1$ and $|\tau_k| < \infty$ in the isotropic one $\epsilon_k = 0$, so that Λ is well-defined.

The Light Cone Singularity

In the case when $\{\hat{\mathbf{c}}_k\} \in \mathbf{c}_\circ^\perp$ for some null vector $\mathbf{c}_\circ \in \mathbb{R}^{2,1}$, $\Lambda(\mathbf{c})$ is decomposable iff $\mathbf{c} \in \mathbf{c}_\circ^\perp$ and the solutions are given by

$$\tau_1 = \frac{(\hat{\mathbf{c}}_2 \curlywedge \mathbf{n})^\circ \tau}{\upsilon_2 \hat{\mathbf{c}}_1^\circ \tau - g_{12} \mathbf{n}^\circ \tau - (\hat{\mathbf{c}}_1 \curlywedge \hat{\mathbf{c}}_2)^\circ}, \quad \tau_2 = \frac{(\hat{\mathbf{c}}_1 \curlywedge \mathbf{n})^\circ \tau}{(\hat{\mathbf{c}}_1 \curlywedge \hat{\mathbf{c}}_2)^\circ + g_{12} \mathbf{n}^\circ \tau - \upsilon_1 \hat{\mathbf{c}}_2^\circ \tau}$$

for the case of two axes and respectively, by the one-parameter set

$$\tau_1 = \frac{\left(\sigma_{32} + (\upsilon_3 \hat{\mathbf{c}}_2^\circ - g_{23} \mathbf{n}^\circ)\tau\right) \tau_2 - \kappa_3 \tau}{\left(g_{13} \hat{\mathbf{c}}_2^\circ - g_{23} \hat{\mathbf{c}}_1^\circ + (\sigma_{13} \upsilon_2 - \sigma_{23} \upsilon_1 + g_{12} \kappa_3)\tau\right) \tau_2 - (\upsilon_3 \hat{\mathbf{c}}_1^\circ - g_{13} \mathbf{n}^\circ)\tau + \sigma_{13}}$$

$$\tau_3 = \frac{\left(\sigma_{12} - (\upsilon_1 \hat{\mathbf{c}}_2^\circ - g_{12} \mathbf{n}^\circ)\tau\right) \tau_2 - \kappa_1 \tau}{\left(g_{12} \hat{\mathbf{c}}_3^\circ - g_{13} \hat{\mathbf{c}}_2^\circ + (\sigma_{12} \upsilon_3 - \sigma_{13} \upsilon_2 + g_{23} \kappa_1)\tau\right) \tau_2 + (\upsilon_1 \hat{\mathbf{c}}_3^\circ - g_{13} \mathbf{n}^\circ)\tau + \sigma_{31}}$$

for the three-axes case, where we denote $\mathbf{x}^\circ = (\mathbf{x}, \mathbf{c}_\circ)\ \forall\, \mathbf{x} \in \mathbb{R}^{2,1}$ as well as

$$\upsilon_k = (\mathbf{n}, \hat{\mathbf{c}}_k), \qquad \sigma_{ij} = (\hat{\mathbf{c}}_i \curlywedge \hat{\mathbf{c}}_j)^\circ, \qquad \kappa_i = (\hat{\mathbf{c}}_i \curlywedge \mathbf{n})^\circ.$$

Change of Coordinates

Bryan to Iwasawa parameters:

$$\theta = 2\,\text{arctan}\,\frac{\sin\tilde{\phi}(\operatorname{ch}\tilde{\vartheta} - \operatorname{sh}\tilde{\vartheta}\operatorname{ch}\tilde{\psi}) - \cos\tilde{\phi}\operatorname{sh}\tilde{\psi}}{\cos\tilde{\phi}(\operatorname{ch}\tilde{\vartheta} - \operatorname{sh}\tilde{\vartheta}\operatorname{ch}\tilde{\psi}) + \sin\tilde{\phi}\operatorname{sh}\tilde{\psi} + \operatorname{ch}\tilde{\vartheta}\operatorname{ch}\tilde{\psi} - \operatorname{sh}\tilde{\vartheta}}$$

$$\beta = 2\,\text{arcth}\,\frac{1 + \operatorname{sh}\tilde{\vartheta} - \operatorname{ch}\tilde{\vartheta}\operatorname{ch}\tilde{\psi}}{1 - \operatorname{sh}\tilde{\vartheta} + \operatorname{ch}\tilde{\vartheta}\operatorname{ch}\tilde{\psi}}, \qquad \lambda = \frac{\operatorname{sh}\tilde{\psi}}{\operatorname{ch}\tilde{\psi} - \operatorname{th}\tilde{\vartheta}}.$$

Iwasawa to Bryan parameters:

$$\tilde{\phi}^{\pm} = 2\,\text{arctan}\,\frac{2\lambda e^{\beta}\cos\theta + (e^{2\beta} + 1 - \lambda^2)\sin\theta}{(e^{2\beta} + 1 - \lambda^2)\cos\theta - 2\lambda e^{\beta}\sin\theta \mp \sqrt{D}}$$

$$\tilde{\vartheta}^{\pm} = 2\,\text{arcth}\,\frac{\lambda^2 + e^{2\beta} - 1}{2e^{\beta} \pm \sqrt{D}}, \qquad \tilde{\psi}^{\pm} = 2\,\text{arcth}\,\frac{2\,\lambda}{\lambda^2 + e^{2\beta} + 1 \pm \sqrt{D}}$$

with the notation $D = \lambda^4 + 2\lambda^2(e^{2\beta} - 1) + (e^{2\beta} + 1)^2$.

A Lift to the Spin Cover

The projective Rodrigues' vector allows for a double-valued lift

$$\zeta_\circ^\pm = \pm(1+\mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_\circ^\pm \mathbf{c}$$

and thus, all results obtained for SO(3) can be extended to SU(2), e.g.

$$\tau_i^\pm = \frac{\sigma_i}{\omega_i \pm \sqrt{\Delta}}, \qquad \xi_k = \pm \frac{1}{\sqrt{1+\tau_k^2}}\left(1+\tau_k \hat{\xi}_k\right).$$

Similarly, in the hyperbolic case one has

$$\zeta_\circ^\pm = \pm(1-\mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^\pm = \zeta_\circ^\pm \mathbf{c}$$

and thus, the decomposition is given as

$$\tau_i^\pm = \frac{\kappa_i}{\omega_i \mp \sqrt{\Delta}}, \qquad \xi_k^\pm = \pm \frac{1}{\sqrt{1-\epsilon_k\tau_k^2}}\left(1+\tau_k \hat{\xi}_k\right).$$

Hyperbolic Geometry and Quantum Scattering

The *monodromy* matrix in scattering theory

$$\mathcal{M} = \frac{1}{t}\begin{pmatrix} 1 & -\bar{r} \\ -r & 1 \end{pmatrix} \in \mathrm{SU}(1,1)$$

may be decomposed in various ways, e.g. as

$$\mathcal{M} = \frac{1}{t}\begin{pmatrix} \mathrm{e}^{\mathrm{i}(\pi-\arg r)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\arg r-\pi)} \end{pmatrix}\begin{pmatrix} 1 & -|r| \\ -|r| & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}(\arg r-\pi)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\pi-\arg r)} \end{pmatrix}.$$

The composition of two pure reflectors yields a phase factor

$$\vartheta = 2\arg(1 + r_1\bar{r}_2)$$

known as Wigner's rotation and respectively, Thomas precession:

$$\tau_\vartheta = \Im\int\limits_\gamma \frac{r\,\mathrm{d}\bar{r}}{1+|r|^2}.$$

Rational Space-Time

Euler-type decompositions in rational Euclidean and hyperbolic 3-spaces:

$$\tau_k^{\pm} = \frac{\sigma^k}{\omega_k \pm \sqrt{\Delta}}, \qquad \sigma^k = \varepsilon^{ijk}(g_{ij} - r_{ij}), \qquad i > j.$$

Pythagorean relations in the Davenport setting

$$\Delta = 1 - r_{31}^2.$$

Rational points on the hyperboloid $\rightarrow$ ultra-hyperbolic quadruples:

$$a^2 + b^2 = c^2 + d^2$$

Iwasawa decomposition in SO(2, 1) yields

$$\tau_1 = \frac{\Lambda_{12} - \Lambda_{32}}{\Lambda_{31} + \Lambda_{13} - \Lambda_{11} - \Lambda_{33}}, \qquad \tau_2 = \frac{1 + \Lambda_{13} - \Lambda_{33}}{1 - \Lambda_{13} + \Lambda_{33}}, \qquad \tau_3 = \frac{\Lambda_{23}}{2\,(\Lambda_{13} - \Lambda_{33})}.$$

Recommended Readings

- Kuvshinov V., Tho N., *Local Vector Parameters of Groups, The Cartan Form and Applications to Gauge and Chiral Field Theory*, Physics of Elementary Particles and the Nucleus **25** (1994).
- Lévay P., *The Geometry of Entanglement*: *Metrics, Connections and the Geometric Phase*, J. Phys. A: Math. & Gen. **37** (2004).
- Brezov D., Mladenova C. and Mladenov I., *Vector-parameters in Classical Hyperbolic Geometry*, J. Geom. Symmetry Phys. **30** (2013).
- Brezov D., Mladenova C. and Mladenov I., *The Geometry of Pythagorean Quadruples and Rational Decomposition of Pseudo-Rotations*, In: Mathematics in Industry, Cambridge Scholars Publishing, Newcastle upon Tyne 2014.
- Brezov D., Mladenova C. and Mladenov I., *Factorizations in Special Relativity and Quantum Scattering on the Line*, In: Advanced Computing in Industrial Mathematics **681**, Springer, Berlin 2017.

The Group SO(4)

Note that $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and consider the representation

$$\mathbb{R}^4 \ni \mathbf{x} \to \mathbf{X} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} + x_4, \quad \det \mathbf{X} = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

which allows for an explicit isometry

$$\mathbf{X} \to \zeta\, \mathbf{X}\, \tilde{\zeta}^{-1}, \qquad \zeta, \tilde{\zeta} \in \mathrm{SU}(2).$$

Introducing the vector-parameters $\mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_\circ}$ and $\tilde{\mathbf{c}} = \dfrac{\tilde{\boldsymbol{\zeta}}}{\zeta_\circ}$, one obtains

$$\mathcal{R}(\mathbf{c} \otimes \tilde{\mathbf{c}}) = \lambda^{-1} \begin{pmatrix} 1 - (\mathbf{c}, \tilde{\mathbf{c}}) + \mathbf{c}\tilde{\mathbf{c}}^t + \tilde{\mathbf{c}}\mathbf{c}^t + (\mathbf{c} + \tilde{\mathbf{c}})^\times & \mathbf{c} - \tilde{\mathbf{c}} + \tilde{\mathbf{c}} \times \mathbf{c} \\ (\tilde{\mathbf{c}} - \mathbf{c} + \tilde{\mathbf{c}} \times \mathbf{c})^t & 1 + (\mathbf{c}, \tilde{\mathbf{c}}) \end{pmatrix}$$

with $\lambda = \sqrt{(1 + \mathbf{c}^2)(1 + \tilde{\mathbf{c}}^2)}$.

...

Conversely, given a rotation matrix $\mathcal{R} \in \mathrm{SO}(4)$, we easily derive

$$\mathbf{c} = \frac{1}{\mathrm{tr}\mathcal{R}} \begin{pmatrix} \tilde{\mathcal{R}}_{32} + \tilde{\mathcal{R}}_{14} \\ \tilde{\mathcal{R}}_{13} + \tilde{\mathcal{R}}_{24} \\ \tilde{\mathcal{R}}_{21} + \tilde{\mathcal{R}}_{34} \end{pmatrix}, \qquad \tilde{\mathbf{c}} = \frac{1}{\mathrm{tr}\mathcal{R}} \begin{pmatrix} \tilde{\mathcal{R}}_{32} - \tilde{\mathcal{R}}_{14} \\ \tilde{\mathcal{R}}_{13} - \tilde{\mathcal{R}}_{24} \\ \tilde{\mathcal{R}}_{21} - \tilde{\mathcal{R}}_{34} \end{pmatrix}$$

where the notation $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}^t$ is used.

The Group SO(2, 2)

Using the isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ and denoting $\lambda = \sqrt{(1-\mathbf{c}^2)(1-\tilde{\mathbf{c}}^2)}$ we obtain

$$\Lambda(\mathbf{c}\otimes\tilde{\mathbf{c}}) = \lambda^{-1}\begin{pmatrix} 1+\mathbf{c}\cdot\tilde{\mathbf{c}} - \mathbf{c}\tilde{\mathbf{c}}^T - \tilde{\mathbf{c}}\mathbf{c}^T + (\mathbf{c}+\tilde{\mathbf{c}})^{\curlywedge} & \mathbf{c}-\tilde{\mathbf{c}}+\tilde{\mathbf{c}}\curlywedge\mathbf{c} \\ (\mathbf{c}-\tilde{\mathbf{c}}-\tilde{\mathbf{c}}\curlywedge\mathbf{c})^T & 1-\mathbf{c}\cdot\tilde{\mathbf{c}} \end{pmatrix}$$

With the notation

$$\tilde{\Lambda} = \Lambda - \Lambda^T = \Lambda - \tilde{\eta}\,\Lambda^t\,\tilde{\eta}^{-1}, \qquad \tilde{\eta} = \mathrm{diag}(1,1,-1,-1)$$

we obtain the vector-parameter for a given pseudo-rotation as

$$\mathbf{c} = \frac{1}{\mathrm{tr}\Lambda}\begin{pmatrix} \tilde{\Lambda}_{14} - \tilde{\Lambda}_{32} \\ \tilde{\Lambda}_{24} + \tilde{\Lambda}_{13} \\ \tilde{\Lambda}_{34} + \tilde{\Lambda}_{21} \end{pmatrix}, \qquad \tilde{\mathbf{c}} = -\frac{1}{\mathrm{tr}\Lambda}\begin{pmatrix} \tilde{\Lambda}_{14} + \tilde{\Lambda}_{32} \\ \tilde{\Lambda}_{24} - \tilde{\Lambda}_{13} \\ \tilde{\Lambda}_{34} + \tilde{\Lambda}_{12} \end{pmatrix}$$

which gives the solution based on the ones in the three-dimensional case.

The Group $\mathrm{SO}^*(4)$

This is the symmetry group of the complex quadric

$$\omega(\mathbf{x},\bar{\mathbf{x}}) = (\mathbf{x}\wedge\bar{\mathbf{x}})_{31} - (\mathbf{x}\wedge\bar{\mathbf{x}})_{42}, \qquad \mathbf{x}\in\mathbb{C}^4.$$

Its Lie algebra is $\mathfrak{so}^*(4) \cong \mathfrak{so}(3)\oplus\mathfrak{sl}(2,\mathbb{R})$ and the block-matrix form is

$$\mathrm{W}(\mathbf{c}\otimes\tilde{\mathbf{c}}) = \begin{pmatrix} a\,\zeta & b\,\zeta \\ c\,\zeta & d\,\zeta \end{pmatrix}$$

where $a,b,c,d\in\mathbb{R}$ with $ad-bc=1$ and $\zeta\in\mathrm{SU}(2)$. This yields

$$\mathbf{c} = \frac{-\,\mathrm{i}}{W_{11}+W_{22}}\begin{pmatrix} W_{11}-W_{22} \\ \mathrm{i}\,W_{12}-\mathrm{i}\,W_{21} \\ W_{12}+W_{21} \end{pmatrix}, \qquad \tilde{\mathbf{c}} = \frac{1}{a+d}\begin{pmatrix} b+c \\ a-d \\ b-c \end{pmatrix}.$$

The Invariant Axis Problem

Two distinct problems:

- for $n = 2k$ invariant axes do not exist in general (Euler)
- for $n > 3$ (pseudo-)rotations are not restricted to a plane (Plücker)

In SO(4) and SO(2, 2) we have such an axis (and plane) if and only if

$$\boldsymbol{\alpha}_+ \perp \boldsymbol{\alpha}_-, \qquad \boldsymbol{\alpha}_\pm = \mathbf{c} \pm \tilde{\mathbf{c}}.$$

We shall address this problem more thoroughly in the next lecture...

Thank You!

THANKS FOR YOUR PATIENCE!