Projective Bivector Parametrization of Isometries
Part III: Clifford's Perspective

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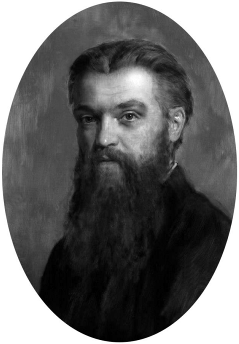
Return of the Jedi ...



W. Clifford [1876]

J. Plücker [1860's]

H. Lorentz [1902]

Recommended Readings

- Brezov D., *Higher-Dimensional Representations of SL_2 and its Real Forms via Plücker Embedding*, Adv. Appl. Clifford Algebras (2017).
- Brezov D., Mladenova C. and Mladenov I., *Wigner Rotation and Thomas Precession: Geometric Phases and Related Physical Theories*, Journal of the Korean Physical Society **66** (2015).
- Kuvshinov V., Tho N., *Local Vector Parameters of Groups, The Cartan Form and Applications to Gauge and Chiral Field Theory*, Physics of Elementary Particles and the Nucleus **25** (1994).
- R. Ward and R. Wells, *Twistor Geometry and Field Theory*, Cambridge University Press, Cambridge 1990.
- A. Bogush and F. Fedorov, *On Plane Orthogonal Transformations* (in Russian), Reports AS USSR **206** (1972) 1033-1036.
- Fedorov F., *The Lorentz Group* (in Russian), Science, Moscow 1979.

Projective Bivectors

Why is **c** not a vector?

- it comes from the bivector part of the quaternion
- it may be infinite (e.g. in the case of half-turns)

The proper term would thus be "projective bivector" and one has formally

$$\mathbf{c} = \frac{\langle\zeta\rangle_2}{\langle\zeta\rangle_0}, \qquad \zeta \in \mathcal{C}\ell^{\circ}_{0,3}/\{0\} \cong \mathbb{H}^*$$

where $\langle\cdot\rangle_k$ denotes grade projection. As for the composition law, one has

$$\mathbf{c}_m \ldots \mathbf{c}_2 \mathbf{c}_1 = \frac{\langle\zeta_k \ldots \zeta_2\zeta_1\rangle_2}{\langle\zeta_k \ldots \zeta_2\zeta_1\rangle_0}.$$

How Far Can We Go?

In direct analogy with the case $n = 3$ one may define

$$\mathbf{c} = \langle \zeta \rangle_0^{-1} \sum_{k=1}^{[\frac{n}{2}]} \langle \zeta \rangle_{2k}, \qquad \zeta \in \mathcal{C}\ell^{\circ}_{p,q} / \{0\}, \qquad p + q = n$$

with $p + q = n$. Furthermore, we still have the composition law

$$\mathbf{c}_m \dots \mathbf{c}_2 \mathbf{c}_1 = \langle \zeta_m \dots \zeta_2 \zeta_1 \rangle_0^{-1} \sum_{k=1}^{[\frac{n}{2}]} \langle \zeta_m \dots \zeta_2 \zeta_1 \rangle_{2k}$$

and the Cayley transform maps this to the usual matrix representation

$$\mathrm{Cay}: \quad \mathrm{P}\mathcal{C}\ell^{\circ}_{p,q} \quad \longrightarrow \quad SO(p, q).$$

However, along the way we lost both homogeneity and decomposability...

The Plücker Embedding

Defining k-blades in $C\ell_n(\mathbb{C})$ as decomposable elements

$$\theta \in \mathcal{B}_k^n \quad \Longleftrightarrow \quad \theta = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k, \quad \mathbf{u}_j \in \mathbb{C}^n$$

one may construct the Plücker embedding as

$$\mathrm{G}_k^n \cong \mathcal{B}_k^n/_{\mathbb{C}^*} \quad \xrightarrow{\mathrm{pl}} \quad \mathrm{P}\bigwedge^k(\mathbb{C}^n), \qquad \mathbb{C}^* \cong \mathbb{C}/\{0\}.$$

In the particular case of bivectors it reduces to the intersection of quadrics

$$\theta \wedge \theta = 0 \quad \rightarrow \quad \theta^{[ij}\theta^{k]l} = 0$$

that yields the embedding of planar (pseudo-)rotations in $\mathrm{SO}(p,q)$, i.e.,

$$\theta_i \wedge \theta_j = 0 \quad \longrightarrow \quad \mathrm{SO}_3 \subset \mathrm{SO}_n.$$

Twistorial Approach

We use double fibrations known from AG and twistors

$$\mathrm{G}_1^n \xleftarrow{\mu} \mathcal{F}_{1,2}^n \xrightarrow{\nu} \mathrm{G}_2^n, \qquad \mathrm{G}_3^n \xleftarrow{\mu^*} \mathcal{F}_{2,3}^n \xrightarrow{\nu^*} \mathrm{G}_2^n$$

to describe the inclusion $\mathfrak{sl}_2 \subset \mathfrak{so}_n$ via incidence relations, e.g. in $\mathbb{C}^4$

$$\mathcal{P}_\alpha \xrightarrow{\rho^{-1}} \ell \xrightarrow{\perp} V_\beta \xrightarrow{\rho_*} \mathcal{P}_\beta .$$

Having determined the invariant direction (in matrix terms) as

$$\ell = \Sigma_1 \cap \Sigma_2, \qquad \Sigma_{1,2} = \{\ker \Theta_{1,2}\} = \{\Theta_{1,2}\}^\perp$$

one may use the commutator and Killing form to write

$$\langle \Theta_2, \Theta_1 \rangle = \frac{\Theta_1 + \Theta_2 + [\Theta_2, \Theta_1]}{1 - (\Theta_1, \Theta_2)} .$$

One Example

Consider the decomposable $\mathrm{SO}^+(4,1)$ element

$$\Theta = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{Cay}} \Lambda = \frac{1}{7}\begin{pmatrix} 1 & -8 & -2 & 4 & -6 \\ -4 & -3 & -6 & -2 & -4 \\ 2 & -2 & 3 & -6 & 2 \\ 8 & 6 & -2 & -3 & 8 \\ 6 & 8 & 2 & -4 & 13 \end{pmatrix}$$

and choose the plane $\{\theta'\} = \{(1,1,0,-1,1)^t, (0,2,1,2,0)^t\}$ and define

$$\ell = \{(1,0,0,0,-1)^t, (2,-1,0,1,0)^t\} \perp \{\theta, \theta'\}.$$

Choosing a basis in the form

$$\mathbf{a}_1 = \mathbf{e}_3, \qquad \mathbf{a}_2 = \mathbf{e}_2 + \mathbf{e}_4, \qquad \mathbf{a}_3 = \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_5$$

in $V_\beta = \ell^\perp$ we perform Bryan decomposition with scalar parameters

$$\tau_1^\pm = \frac{4}{9 \pm \sqrt{33}}, \qquad \tau_2^\pm = \frac{2}{-7 \pm \sqrt{33}}, \qquad \tau_1^\pm = \frac{2}{3 \mp \sqrt{33}}.$$

The Proper Lorentz Group $SO^+(3,1)$

Consider the isomorphism

$$\mathcal{C}\ell_{0,3} \cong \mathcal{C}\ell^{\circ}_{1,3} \cong \mathbb{H}^{\mathbb{C}}$$

that yields on the Lie group level

$$\mathrm{SO}^+(3,1) \cong \mathrm{SO}(3,\mathbb{C})$$

and we have the matrix realization (Fedorov)

$$\Lambda(\mathbf{c}) = \frac{1}{|1+\mathbf{c}^2|}\begin{pmatrix} 1-|\,\mathbf{c}\,|^2 + \mathbf{c}\bar{\mathbf{c}}^t + \bar{\mathbf{c}}\mathbf{c}^t + (\mathbf{c}+\bar{\mathbf{c}})^{\times} & \mathrm{i}(\bar{\mathbf{c}} - \mathbf{c} + \bar{\mathbf{c}} \times \mathbf{c}) \\ \mathrm{i}(\bar{\mathbf{c}} - \mathbf{c} - \bar{\mathbf{c}} \times \mathbf{c})^t & 1 + |\,\mathbf{c}\,|^2 \end{pmatrix}$$

where $\mathbf{c} = \boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta} \in \mathbb{CP}^3$ is the complex vector parameter. Denoting $\tilde{\Lambda} = \Lambda - \tilde{\eta}\,\Lambda^t\,\tilde{\eta}$, where $\tilde{\eta} = \mathrm{diag}(1,1,1,-1)$, we derive its components as

$$\boldsymbol{\alpha} = \frac{1}{\mathrm{tr}\Lambda}\left(\tilde{\Lambda}_{32},\, \tilde{\Lambda}_{13},\, \tilde{\Lambda}_{21}\right)^t, \qquad \boldsymbol{\beta} = \frac{1}{\mathrm{tr}\Lambda}\left(\tilde{\Lambda}_{14},\, \tilde{\Lambda}_{24},\, \tilde{\Lambda}_{34}\right)^t.$$

Wigner Rotation and Thomas Precession

The *Wigner* angle in 3D relativity is defined as

$$\theta = 2 \arg\left(1 + \bar{z}_1 z_2\right)$$

where $z_k \in \mathbb{C}$ are the stereographic projections of the two boosts' vector-parameters. On the infinitesimal level (in the Thomas frame)

$$\mathrm{d}\tau_\theta = -\Im \frac{\bar{z}\,\mathrm{d}z}{1 - |z|^2}.$$

Adding the Euclidean case (Foucault's pendulum) Stoke's theorem yields

$$\omega_h = -\Im \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{(1 - |z|^2)^2}, \qquad \omega_e = \Im \frac{\mathrm{d}\bar{z} \wedge \mathrm{d}z}{(1 + |z|^2)^2}$$

given by the Fubini-Study construction for the Hopf bundles

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2, \qquad \mathbb{S}^1 \rightarrow \mathrm{AdS}_3 \rightarrow \mathbb{D}.$$

Electrodynamics and Beyond

We extend the complex representation of the EM field to

$$\mathbf{c} = \boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta} \in \mathbb{CP}^3.$$

In $\mathbb{R}^{3,1}$ boosts are represented by imaginary bivectors $\mathbf{c} = \mathrm{i}\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{B}^3$ that may be mapped to $\zeta \in \mathbb{H}$ leading to the expression for the EM induction

$$\Re\langle \mathrm{i}\boldsymbol{\beta}_2, \mathrm{i}\boldsymbol{\beta}_1 \rangle = \frac{\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2}{1 + (\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)} \quad \rightarrow \quad \mathcal{A} = \Im \frac{\bar{\zeta}\, \mathrm{d}\zeta}{1 + |\zeta|^2}$$

while in the compact case there is no holonomy as the bundle

$$\mathrm{SO}(4) \longrightarrow \mathrm{SO}(3), \qquad \mathrm{Spin}(4) \cong \mathrm{Spin}(3) \otimes \mathrm{Spin}(3)$$

is globally trivial. In higher dimensions (if the Plücker relations hold) we use similar technique to express the corresponding geometric phases as

$$\mathrm{d}\Theta_W^\circ = \frac{[\Theta, \mathrm{d}\Theta\,]}{1 + ||\Theta\,||^2}, \qquad \mathrm{d}\tilde{\Theta}_W^\circ = \frac{[\tilde{\Theta}, \mathrm{d}\tilde{\Theta}\,]}{1 - ||\tilde{\Theta}\,||^2}.$$

Invariant Axes and Wigner's Little Groups

The Plücker relations may be written for the vector-parameter as

$$\Im\, \mathbf{c}^2 = 0 \quad \Leftrightarrow \quad \boldsymbol{\alpha} \perp \boldsymbol{\beta}$$

in which case the fixed subspace of $\Lambda(\mathbf{c})$ is spanned by

$$\boldsymbol{\sigma}_1 = (\boldsymbol{\alpha},\, 0)^t, \qquad \boldsymbol{\sigma}_2 = (\boldsymbol{\alpha} \times \boldsymbol{\beta},\, \boldsymbol{\alpha}^2)^t.$$

The corresponding Wigner little groups are related to the bundles

$$\mathbb{B}_3 \cong \mathrm{SO}^+(3,1)/\mathrm{SO}(3), \qquad \mathrm{dS}_3 \cong \mathrm{SO}^+(3,1)/\mathrm{SO}(2,1)$$

$$\mathcal{L}(\mathbb{R}^{3,1}) \cong \mathrm{SO}^+(3,1)/\mathrm{E}(2)$$

used describe elementary particles (bradyons, tachyons and luxons).

Alternative Parameterizations

Consider the vector-parameter

$$\mathbf{c} = (3 + 2\mathrm{i},\, 3\mathrm{i} - 2,\, 2 + \mathrm{i})^t$$

and determine the two characteristic directions

$$\mathbf{c}_\circ = (1,\, \mathrm{i},\, 0)^t \in \ker\left(\mathbf{c}^\times \pm \mathrm{i}\sqrt{\mathbf{c}^2}\right), \qquad \boldsymbol{\kappa} = (0,\, 0,\, 1)^t = |\boldsymbol{\alpha}_\circ|^{-2}\boldsymbol{\alpha}_\circ \times \boldsymbol{\beta}_\circ$$

that allow for a factorization

$$\begin{aligned} \mathbf{c} &= \langle (1 - 3\mathrm{i}/2)\,\mathbf{c}_\circ,\, \mathrm{i}(3 \mp 2\sqrt{2})\,\boldsymbol{\kappa},\, (1 \pm \sqrt{2})\,\boldsymbol{\kappa}\rangle \\ \mathbf{c} &= \langle (1 \pm \sqrt{2})\,\boldsymbol{\kappa},\, \mathrm{i}(3 \mp 2\sqrt{2})\,\boldsymbol{\kappa},\, (1/4 + 5\mathrm{i}/4)\,\mathbf{c}_\circ\rangle. \end{aligned}$$

One may also decompose into mutually commuting boosts and rotations

$$\Lambda = \Lambda_3\mathcal{R}_3\Lambda_2\mathcal{R}_2\Lambda_1\mathcal{R}_1 = \mathcal{R}_2\Lambda_2\mathcal{R}_2\Lambda_2\mathcal{R}_1\Lambda_1.$$

The Dual Extension

We consider a central extension to a given algebra

$$x \to \underline{x} = x + \varepsilon t, \qquad \varepsilon^2 = 0$$

that clearly yields for analytic functions

$$f(x + \varepsilon t) = f(x) + \varepsilon f'(x) t.$$

In particular, one may have dual quaternion or axis-angle variables

$$\underline{\mathbf{n}} = \mathbf{n} + \varepsilon \mathbf{m}, \qquad \underline{\varphi} = \varphi + \varepsilon \psi$$

that leads to the dual Rodrigues' vector

$$\underline{\mathbf{c}} = \left(\tau + (1 + \tau^2) \frac{\psi}{2} \varepsilon \right) \underline{\mathbf{n}}.$$

Recommended Readings

- Chub V., *On the Possibility of Application of One System of Hypercomplex Numbers in Inertial Navigation*, Mech. Solids **37** (2002).
- Condurache D. and Burlacu A., *Dual Tensor Based Solutions for Rigid Body Motion Parameterization*, Mehcanisms and Machine Theory **74** (2014).
- Wittenburg J., *Kinematics: Theory and Applications*, Springer Verlag Berlin Heidelberg 2016
- Dimentberg, F. *The Screw Calculus and Its Applications in Mechanics*, Foreign Technology Division (1965).

Homework:

What should be done now?

- Tell your friends about what you've learned!
- Tell them to tell their friends!
- Do some research and see how easy it is!
- Don't forget to cite our papers!

Thank You!

THANKS FOR YOUR PATIENCE!