On Hypercomplex Calculi with Kinematical Origins

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The Cross Product

Consider the Hodge star operator defined in Clifford basis as

$$\star : \quad \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \dots \mathbf{e}_k \;\to\; \mathbf{e}_{k+1} \wedge \mathbf{e}_{k+2} \wedge \dots \mathbf{e}_n$$

and use it to construct the Cross product in $\mathbb{R}^3$:

$$\mathbf{u} \times \mathbf{v} = \star(\mathbf{u} \wedge \mathbf{v})$$

that clearly yields a map $\alpha : \mathbb{R}^3 \to \mathrm{End}(\mathbb{R}^3)$ in the form

$$\alpha : \quad \mathbf{u} \;\to\; \hat{\mathbf{u}} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \;\in\; \mathrm{End}(\mathbb{R}^3).$$

Kinematical Context

In rigid body one has the constraint $\frac{d}{dt}\mathbf{r}^2 = 0$ and thus

$$\dot{\mathbf{r}} = \hat{\omega}\,\mathbf{r}, \qquad \hat{\omega} = \dot{\mathcal{R}}\mathcal{R}^t \in \mathfrak{so}(3)$$

In the simple case $\boldsymbol{\omega} = \mathrm{const.}$, the solution has the form

$$\mathbf{r}(t) = \mathrm{e}^{t\hat{\omega}}\mathbf{r}_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!}\,\hat{\omega}^k\mathbf{r}_0, \qquad \mathbf{r}_0 = \mathbf{r}(0)$$

Similarly, one has Euler's dynamical equations

$$\dot{\mathbf{L}} = -\hat{\omega}\,\mathbf{L} + \mathbf{M}, \qquad \mathbf{L} = \mathrm{I}\,\boldsymbol{\omega}.$$

Iterations

Homogeneity allows for a restriction to the unit sphere

$$\mathbb{R}^3 \ni \mathbf{x} = \lambda \boldsymbol{\xi}, \qquad \boldsymbol{\xi} \in \mathbb{S}^2, \qquad \lambda = ||\mathbf{x}|| \in \mathbb{R}^+$$

and thus expressing (for $n \geq 0$)

$$\hat{\boldsymbol{\xi}}^{n+2} = -\hat{\boldsymbol{\xi}}^n, \qquad \hat{\boldsymbol{\xi}}^0 = \mathcal{I}$$

where we use the standard notation for the projectors

$$\mathcal{P}_\xi^{||} = \boldsymbol{\xi}\boldsymbol{\xi}^t, \qquad \mathcal{P}_\xi^{\perp} = \mathcal{I} - \mathcal{P}_\xi^{||}.$$

It is not hard to show by induction that

$$\hat{\boldsymbol{\xi}}_{2k+1} \dots \hat{\boldsymbol{\xi}}_2 \hat{\boldsymbol{\xi}}_1 = (-1)^k g_{2k+1[2k} \cdots g_{3[2} \boldsymbol{\xi}_{1]]\dots]}$$

where we denote $g_{ij} = \hat{\boldsymbol{\xi}}_i \cdot \hat{\boldsymbol{\xi}}_j$ and $a_{[i} b_{j]} = a_i b_j - a_j b_i$.

Algebraic Construction

Consider the hypercomplex number system

$$\Omega : \quad \{p,\, q,\, r\} \quad \longleftrightarrow \quad \{\mathcal{P}^{\|}_{\xi},\, \mathcal{P}^{\perp}_{\xi},\, \hat{\boldsymbol{\xi}}\}$$

defined by the multiplications

$$p^2 = p, \qquad p\,q = p\,r = 0, \qquad qr = r, \qquad q^2 = -r^2 = q$$

that clearly indicate the isomorphism $\Omega \cong \mathbb{R} \oplus \mathbb{C}$. Hence, one has

$$\varphi = \varphi_0 p + \varphi_1 q + \varphi_2 r \quad \longrightarrow \quad \{\varphi_0,\, \varphi_1 + i\varphi_2\} \in \mathbb{R} \oplus \mathbb{C}$$

and Ω inherits its properties from the real and complex algebra.

Cylindrical Representation

Consider the projections

$$\langle\varphi\rangle_0 = p\,\varphi, \qquad \langle\varphi\rangle_\perp = q\varphi$$

allowing us to consider separate norms in Ω_0 and $\Omega_\perp$. Then

$$\varphi_1 + i\varphi_2 = \rho\,\mathrm{e}^{i\vartheta}, \qquad \rho = ||\varphi||_\perp, \qquad \vartheta = \arg\langle\varphi\rangle_\perp = \mathrm{atan}_2\,\frac{\varphi_2}{\varphi_1}$$

and hence, the famous Moivre's formula

$$\varphi^n = \varphi_0^n\,p + \rho^n\,[\cos{(n\vartheta)}q + \sin{(n\vartheta)}r] = \varphi_0^n\,p + \rho^n\langle\mathrm{e}^{n\vartheta r}\rangle_\perp$$

as well as the formula for the n-th root

$$(\sqrt[n]{\varphi})_{jk} = (\sqrt[n]{\varphi_0})_j\,p + \rho^{\frac{1}{n}}\left(\cos\frac{\vartheta+2k\pi}{n}\,q + \sin\frac{\vartheta+2k\pi}{n}\,r\right).$$

Analyticity and Invertibility

The expansion $\mathrm{End}(\Omega) \ni f(\varphi) = f_0\, p + f_1\, q + f_2\, r$ can be written also as

$$\{\varphi_0,\, z\} \xrightarrow{f} \{f_0(\varphi_0),\, h(z)\}, \qquad h(z) = f_1(z) + i f_2(z)$$

Then, f is analytic in Ω iff f_0, h are analytic respectively in $\mathbb{R}$ and $\mathbb{C}$.

$$\varphi \quad \to \quad \underline{\varphi} = \begin{pmatrix} \varphi_0 & 0 & 0 \\ 0 & \varphi_1 & -\varphi_2 \\ 0 & \varphi_2 & \varphi_1 \end{pmatrix}$$

in a suitable basis and

$$||\varphi|| = |\det \underline{\varphi}| = ||\varphi||_0 ||\varphi||_\perp^2 = |\varphi_0|\,(\varphi_1^2 + \varphi_2^2)$$

so $\exists\, \varphi^{-1} \Leftrightarrow ||\varphi|| \neq 0$ and similarly, if f is analytic $\exists\, f^{-1} \Leftrightarrow ||f'|| \neq 0$.

Some Useful Formulas

Consider the geometric series

$$\sum_{n=0}^{\infty} \varphi^n = \frac{p}{1-\varphi_0} + \frac{(1-\varphi_1)\,q + \varphi_2 r}{(1-\varphi_1)^2 + \varphi_2^2}, \qquad ||\varphi||_0, ||\varphi||_\perp < 1$$

as well as the Cayley transform

$$\mathrm{Cay}(\varphi) = \frac{1+\varphi_0}{1-\varphi_0}\,p + \frac{(1-||\varphi||_\perp^2)\,q + 2\varphi_2 r}{(1-\varphi_1)^2 + \varphi_2^2}$$

and in particular

$$\mathrm{Cay}(\lambda r) = p + \frac{1-\lambda^2}{1+\lambda^2}\,q + \frac{2\lambda}{1+\lambda^2}\,r.$$

The exponent is a typical example of globally analytic map

$$\exp\varphi = \mathrm{e}^{\varphi_0} p + \mathrm{e}^{\varphi_1}\left(\cos\varphi_2\, q + \sin\varphi_2\, r\right), \qquad \exp\varphi \exp\psi = \exp(\varphi + \psi)$$

The Proper Lorentz Group

Similarly, we consider $\mathbb{C}^3 \to End(\mathbb{C}^3)$ and use the isomorphism

$$\mathrm{SO}(3,\mathbb{C}) \cong \mathrm{SO}^+(3,1)$$

to construct the Lorentz equivalent. Note also that

$$\hat{\mathbf{x}}^2 = \mathbf{x} \otimes \mathbf{x} - \mathbf{x}^2 \mathcal{I}$$

and as long as $\mathbf{x}^2 \neq 0$, one can normalize as

$$\mathbf{x} = \lambda \boldsymbol{\xi}$$

with $\boldsymbol{\xi}^2 = 1$ and $\lambda \in \mathbb{C}$ that yields a complexification of Ω.

Duplex Numbers

For non-isotropic vectors $\mathbf{x}^2 \neq 0$ it is straightforward to show that

$$\Omega^{\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{D}, \qquad \mathbb{D} \cong \mathcal{C}\ell_1(\mathbb{C})$$

where the bicomplex (duplex) numbers $\mathbb{D}$ are generated by $\{1, i, j, k\}$ with $i^2 = j^2 = -1$, $ij = k$ is shown to be $\mathbb{D} \cong \mathbb{C}^2$ via the idempotents

$$\tau_\pm = \frac{1}{2}(1 \pm k), \qquad \tau_\pm^2 = \tau_\pm, \qquad \tau_+\tau_- = 0$$

that yields the decomposition

$$\psi_\perp = \psi_-\tau_- + \psi_+\tau_+, \qquad \psi_\pm = \psi_1 \mp i\psi_2 \in \mathbb{C}.$$

Bicomplex holomorphic functions satisfy

$$\bar{\partial}\psi = \partial^*\psi = \bar{\partial}^*\psi = 0$$

that may also be written as $D^{(4)}\psi = 0$, $D^{(2)}\psi_{1,2} = 0$.

The Isotropic Case

In the isotropic case $\mathbf{x}^2 = 0$ one has $\hat{\mathbf{x}}^2 = \mathbf{x}\mathbf{x}^t$ and thus

$$\Omega^{\mathbb{C}}_{null} : \quad \{1,\, \ell,\, \epsilon\}, \quad \ell^2 = \epsilon, \quad \ell^3 = 0$$

is isomorphic to the matrix algebra

$$\Omega^{\mathbb{C}}_{null} \ni \psi = \psi_0 + \psi_1 \ell + \psi_2 \epsilon \quad \leftrightarrow \quad \underline{\psi} = \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 \\ 0 & \psi_0 & \psi_1 \\ 0 & 0 & \psi_0 \end{pmatrix}.$$

For example, one has the multiplication rule

$$\varphi\,\psi = \varphi_0\psi_0 + (\psi_0\varphi_1 + \varphi_0\psi_1)\ell + (\varphi_1\psi_1 + \psi_0\varphi_2 + \varphi_0\psi_2)\epsilon$$

and Taylor expansion of functions over this algebra yields

$$f(\psi) = f(\psi_0) + f'(\psi_0)\,[\psi_1\ell + \psi_2\,\epsilon] + \frac{1}{2} f''(\psi_0)\psi_1^2\,\epsilon$$

for example

$$\exp\psi = \mathrm{e}^{\psi_0}\left[1 + \psi_1\ell + \left(\psi_2 + \frac{\psi_1^2}{2}\right)\epsilon\right].$$

Real Forms

- Dual complex numbers are embedded in the even subalgebra

$$\mathbb{C}[\epsilon] \cong \mathrm{E}(2) \subset \Omega^{\mathbb{C}}_{null} : \qquad \{1, \epsilon\}, \qquad \epsilon^2 = 0.$$

- The hyperbolic real form $\Omega \cong \mathbb{R} \oplus \mathbb{C}'$, where

$$\mathbb{C}' \cong \mathcal{C}\ell_{1,0} \cong \mathbb{R}^2 : \qquad \{1, k\}, \qquad j^2 = 1.$$

  The Cauchy-Riemann analyticity conditions in this case are

$$\frac{\partial f_1}{\partial \varphi_1} = \frac{\partial f_2}{\partial \varphi_2}, \qquad \frac{\partial f_1}{\partial \varphi_2} = \frac{\partial f_2}{\partial \varphi_1} \quad \Leftrightarrow \frac{\partial h}{\partial z^*} = 0\cdot$$

  One example is the exponential map

$$\exp(\varphi_1 + k\varphi_2) = \mathrm{e}^{\varphi_1}(\cosh \varphi_2 + k \sinh \varphi_2) = \mathrm{e}^{\varphi_1 - \varphi_2}\tau_- + \mathrm{e}^{\varphi_1 + \varphi_2}\tau_+$$

  where $\tau_\pm = \frac{1}{2}(1 \pm k)$ yield the retarded and accelerated wave.

- The Euclidean real form $\Omega \cong \mathbb{R} \oplus \mathbb{C}$ has already been discussed.

Recommended Readings

- Todorov V., *Analytic Vector Functions*, AIP Conf. Proc. **xxx** (2017).
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- Mladenova C., Brezov D. and Mladenov I., *New Forms of the Equations of the Attitude Kinematics*, PAMM **14** (2014).
- Davenport C. *A Hypercomplex Calculus with Applications to Special Relativity*. Knoxville, Tennessee 1991, ISBN 0962383708.
- Kassandrov V., *Biquaternion Electrodynamics and Weyl-Cartan Geometry of Space-Time*, Gravitat. & Cosmol. **3** (1995).
- Aste A., *Complex Representation Theory of the Electromagnetic Field*, J. Geom. Symmetry Phys. **28** (2012).

Deformations

Consider a smooth flow $t \to g(t)$ on the Ω-bundle over $\mathbb{R}^3$ and

$$\dot{\varphi} = \dot{\varphi}_0 p + \dot{\varphi}_1 q + \dot{\varphi}_2 r + (\varphi_0 - \varphi_1)(\dot{r}r + r\dot{r}) + \varphi_2 \dot{r}$$

using the correspondence

$$\dot{r} \longleftrightarrow \dot{\hat{\xi}}, \qquad \dot{p} = -\dot{q} = \dot{r}r + r\dot{r} \longleftrightarrow \xi\dot{\xi}^t + \dot{\xi}\xi^t.$$

One may consider the non-commutative term ω_f in $\mathrm{d}f \leftrightarrow \{\mathrm{d}f_0, \mathrm{d}h, \omega_f\}$

$$\omega_f = f_{01}\mathrm{d}q + f_2\mathrm{d}r, \qquad f_{01} = f_1 - f_0$$

from the perspective of bundle holonomy and study the geometric phase

$$\oint_\gamma \mathrm{d}f = \oint_\gamma \omega_f, \qquad f: \ \text{analytic}$$

Thank You!

THANKS FOR YOUR PATIENCE!