Quantization of Locally Symmetric Kähler Manifolds

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Section 0

Talk plan

Talk plan

1. The deformation quantization with separation of variables→Our main theorem!
2. One and two dimensional cases →One:trivial but $\cdots$
3. A metric of a complex Grassmann manifold
4. Deformation quantization for complex Grassmann manifold :$\mathbb{C}P^n$:not so hard but $\cdots$

Section 1

The deformation quantization with separation of variables

Locally symmetric Kähler manifold

Definition (Locally symmetric Kähler manifold)

A Riemannian(Kähler) manifold (M, g) is called a locally symmetric Riemannian(Kähler) manifold when

$$\nabla_m R_{ijk}{}^l = 0 \ (\forall i, j, k, l, m).$$

Example (Locally symmetric Kähler manifold)

Examples

- *Symmetric Kähler manifold*
  (Kähler homogeneous space with a transitive action)
- *Compact Riemann surface*
- *Complex Grassmann manifold (including projective space $\mathbb{C}P^n$)*
- *and more $\cdots$*

Deformation quantization

Definition (Deformation quantization of Poisson manifolds)

$\mathcal{F}$ is defined as a set of formal power series:
$\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k f_k \hbar^k,\ f_k \in C^\infty(M) \right\}$. A star product $* : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, is defined as

$$f * g = \sum_k C_k(f, g) \hbar^k$$

such that the product satisfies the following conditions.

1. $(\mathcal{F}, +, *)$ is a (noncommutative) algebra.
2. $C_k(\cdot, \cdot)$ is a bidifferential operator.
3. C_0 and C_1 are defined as

$$C_0(f, g) = fg, \qquad C_1(f, g) - C_1(g, f) = \{f, g\}, \tag{1}$$

   where $\{f, g\}$ is the Poisson bracket.
4. $f * 1 = 1 * f = f$.

A star product with separation of variables

Definition (A star product with separation of variables)

$*$ is called a star product with separation of variables on a Kähler manifold M when

$$a * f = af \, , f * b = fb$$

where $\bar{\partial} a = \partial b = 0$.

Definition (Def of L_f, R_f)

$$L_f g := f * g \qquad , \qquad R_g f := f * g$$

Remark (Associativity$\Longleftrightarrow$Commutativity)

$$(f * g) * h = f * (g * h)$$
$$\Longleftrightarrow R_h (L_f g) = L_f (R_h g) \Longleftrightarrow [L_f, R_h] = 0$$

Our main theorem

We use

$$\mathcal{S} := \left\{ A \mid A = \sum_{\alpha} a_\alpha D^\alpha, \quad a_\alpha \in C^\infty(M) \right\},$$

where $D^{\bar{l}} := g^{\bar{l}k}\partial_k$ and α is a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. In this article, we also use the Einstein summation convention over repeated multi-indices and $a_\alpha D^\alpha := \sum_\alpha a_\alpha D^\alpha$.

Our main theorem

When the star product with separation of variables for smooth functions f and g on a local symmetric Kähler manifold is given as

$$f * g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f \right) \left(D^{\vec{\beta_n^*}} g \right),$$

these smooth functions $T^n_{\vec{\alpha_n}\vec{\beta_n^*}}$, which are covariantly constants, are determined by the following recurrence relations for $\forall i$:

Recurrence relations of our main theorem

$$\sum_{d=1}^{N} \hbar g_{\bar{i}d} T^{n-1}_{\vec{\alpha_n}-\vec{e_d}\vec{\beta_n^*}-\vec{e_i}}$$

$$= \beta_i T^{n}_{\vec{\alpha_n}\vec{\beta_n^*}}$$

$$+ \sum_{k=1}^{N}\sum_{p=1}^{N} \frac{\hbar\left(\beta_k^n - \delta_{kp} - \delta_{ik} + 1\right)\left(\beta_k^n - \delta_{kp} - \delta_{ik} + 2\right)}{2} R_{\bar{p}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\ T^{n}_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_p}+2\vec{e_k}-\vec{e_i}}$$

$$+ \sum_{k=1}^{N-1}\sum_{l=1}^{N-k}\sum_{p=1}^{N} \hbar\left(\beta_k^n - \delta_{kp} - \delta_{ik} + 1\right)\left(\beta_{k+l}^n - \delta_{(k+l),p} - \delta_{i,(k+l)} + 1\right)$$

$$\times R_{\bar{p}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\ T^{n}_{\vec{\alpha_n}\vec{\beta_n^*}-\vec{e_p}+\vec{e_k}+\vec{e_{k+l}}-\vec{e_i}}.$$

Theorem

[Karabegov [15]]. For an arbitrary Kähler form ω, there exist a star product with separation of variables $*$ and it is constructed as follows. Let f be an element of $\mathcal{F}$ and $A_n \in \mathcal{S}$ be a differential operator whose coefficients depend on f i.e.

$$A_n = a_{n,\alpha}(f) D^{\alpha}, \ D^{\alpha} = \prod_{i=1}^{n} (D^{\bar{i}})^{\alpha_i}, \ D^{\bar{i}} = g^{\bar{i}l} \partial_l, \tag{2}$$

where α is an multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then,

$$L_f = \sum_{n=0}^{\infty} \hbar^n A_n \tag{3}$$

is uniquely determined such that it satisfies the following conditions.

Theorem (following conditions)

For $R_{\partial_{\bar{l}}\Phi} = \partial_{\bar{l}}\Phi + \hbar\partial_{\bar{l}}$,

$$\left[L_f, R_{\partial_{\bar{l}}\Phi}\right] = 0 \ , \qquad L_f 1 \ = \ f * 1 = f. \tag{4}$$

Then the star products are given by $L_f g := f * g$ *and the star products satisfy the associativity;*

$$L_h(L_g f) \ = \ h * (g * f) = (h * g) * f = L_{L_h g} f. \tag{5}$$

Recall that each two of $D^{\bar{i}}$ commute each other, so if a multi index α is fixed then the A_n is uniquely determined. (4)-(5) imply that $L_f g = f * g$ gives deformation quantization.

Fact (Transition maps)

The following formulas are given in [30].For $U_a \cap U_b \neq \emptyset$

$$f *_b g\,(w, \bar{w}) = \phi^*_{a,b} f *_a g\,(w, \bar{w}) = \phi^*_{a,b} f\,(w\,(z)\,, \bar{w}\,(\bar{z})) *_a g\,(w\,(z)\,, \bar{w}\,(\bar{z}))$$

We need to solve

Remark

We need to solve

$$\left[L_f, R_{\partial_{\bar{l}}\Phi}\right] = [L_f, \partial_{\bar{l}}\Phi] + [L_f, \hbar\partial_{\bar{l}}] = 0.$$

Definition (Twisted symbol)

A map from differential operators to formal polynomials is defined as

$$\sigma(A;\xi) := \sum_{\alpha} a_\alpha \xi^\alpha,$$

where

$$A = \sum_{\alpha} a_\alpha D^\alpha.$$

This map is called "twisted symbol". It becomes easier to calculate commutators by using the following theorem.

Useful formulae for the first term $[L_f, \partial_{\bar{i}}\Phi]$

Remark

There are some useful formulae. $D^{\bar{l}}$ satisfies the following equations.

$$[D^{\bar{l}}, D^{\bar{m}}] = 0 \ , \ [D^{\bar{l}}, \partial_{\bar{m}}\Phi] = \delta^{\bar{l}}{}_{\bar{m}}, \quad \forall l, m, \tag{6}$$

where $[A,\ B] = AB - BA$ and Φ is a Kähler potential . Using them, one can construct a star product as a differential operator L_f .

Proposition (Karabegov [15])

Let $a(\xi)$ be a twisted symbol of an operator A. Then the twisted symbol of the operator $[A, \partial_{\bar{i}}\Phi]$ is equal to $\partial a/\partial\xi^{\bar{i}}$;

$$\sigma\left([A, \partial_{\bar{i}}\Phi]\right) = \frac{\partial}{\partial\xi^{\bar{i}}}\sigma(A).$$

This proposition follows from From (6), i.e. $\sigma\left([D^{\bar{l}}, \partial_{\bar{i}}\Phi]\right) = \delta^{\bar{l}}_{\bar{i}}$.

The first term $[L_f, \partial_{\bar{i}}\Phi]$

Proposition (The first term)

*Let f and g be smooth functions on a locally symmetric Kähler manifold Mand L_f be a left star product by f given as (**??**). Then*

$$[L_f, \partial_{\bar{i}}\Phi]\, g = \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(D^{\vec{\beta_n^*}-\vec{e_i}} g\right) \tag{7}$$

or equivalently,

$$\sigma\left([L_f, \partial_{\bar{i}}\Phi]\right) = \frac{\partial \sigma\left(L_f\right)}{\partial \xi^{\bar{i}}}$$

$$= \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_i^n T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right) \left(\xi^{\bar{1}^{\beta_1^n}} \cdots \xi^{\bar{i}^{\beta_i^n - 1}} \cdots \xi^{\bar{N}^{\beta_N^n}}\right)$$

The second term $[L_f, \hbar\partial_{\bar{l}}]$

Proposition (The second term)

*Let f and g be smooth functions on a locally symmetric Kähler manifold M. Let L_f be a left star product by f given as (**??**). Then,*

$$
\begin{aligned}
&[L_f, \hbar\partial_{\bar{i}}]g \\
&= \hbar \sum_{n=0}^{\infty} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \sum_{k=1}^{N} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \frac{\beta_k^n(\beta_k^n - 1)}{2} R_{\bar{\rho}}{}^{\bar{k}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right)\left(D^{\vec{\beta_n^*}+\vec{e_\rho}-\vec{e_k}} g\right) \\
&\quad + \hbar \sum_{n=0}^{\infty} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\vec{\alpha_n}\vec{\beta_n^*}} \beta_k^n \beta_{k+l}^n R_{\bar{\rho}}{}^{\overline{k+l}\bar{k}}{}_{\bar{i}}\, T^n_{\vec{\alpha_n}\vec{\beta_n^*}} \left(D^{\vec{\alpha_n}} f\right)\left(D^{\vec{\beta_n^*}+\vec{e_\rho}-\vec{e_k}} g\right) \\
&\quad - \hbar \sum_{n=1}^{\infty} \sum_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \sum_{d=1}^{N} g_{\bar{i}d}\, T^{n-1}_{\vec{\alpha_{n-1}}\vec{\beta_{n-1}^*}} \left(D^{\vec{\alpha_{n-1}}+\vec{e_d}} f\right)\left(D^{\vec{\beta_{n-1}^*}} g\right).
\end{aligned}
$$

The second term $[L_f, \hbar\partial_l]$

The following formulas are given in [28].

Fact ($D \fallingdotseq \nabla$)

For smooth functions f and g on a locally symmetric Kähler manifold, the following formulas are given.

$$
\begin{aligned}
\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f &= g_{l_1\bar{j}_1} \cdots g_{l_n\bar{j}_n} D^{l_1} \cdots D^{l_n} f \\
\nabla_{k_1} \cdots \nabla_{k_n} g &= g_{\bar{m}_1 k_1} \cdots g_{\bar{m}_n k_n} D^{\bar{m}_1} \cdots D^{\bar{m}_n} g \\
D^{l_1} \cdots D^{l_n} f &= g^{l_1\bar{j}_1} \cdots g^{l_n\bar{j}_n} \nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \\
D^{\bar{m}_1} \cdots D^{\bar{m}_n} g &= g^{\bar{m}_1 k_1} \cdots g^{\bar{m}_n k_n} \nabla_{k_1} \cdots \nabla_{k_n} g.
\end{aligned}
$$

If $\beta^n_a > 1$ and $\sum_{k=a+1}^{N} \beta^n_k > 0$, by using Fact 2, we obtain (**??**) as

$$\sum_{m=1}^{\beta^n_a} \left(D^{\bar{a}}\right)^{m-1} \left[D^{\bar{a}}, \nabla_{\bar{i}}\right] \left(D^{\bar{a}}\right)^{\beta^n_a - m} \left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}} \cdots \left(D^{\bar{N}}\right)^{\beta^n_N} g \tag{8}$$

$$= \sum_{m=1}^{\beta^n_a} \left(D^{\bar{a}}\right)^{m-1} g^{\bar{a}b} g^{\bar{a}k_{a,1}} \cdots g^{\bar{a}k_{a,\beta^n_a - m}} [\nabla_b, \nabla_{\bar{i}}] \nabla_{k_{a,1}} \cdots \nabla_{k_{a,\beta^n_a - m}} \left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}} \cdots \left(D^{\bar{N}}\right)^{\beta^n_N} g$$

$$= \sum_{m=1}^{\beta^n_a} \sum_{n_a=1}^{\beta^n_a - m} \left(D^{\bar{a}}\right)^{m-1} R_{\bar{i}}{}^{\bar{a}\bar{a}}{}_{\bar{c}} D^{\bar{c}} \left(D^{\bar{a}}\right)^{\beta^n_a - m} \left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}} \cdots \left(D^{\bar{N}}\right)^{\beta^n_N} g$$

$$+ \sum_{m=1}^{\beta^n_a} \sum_{j=a+1}^{N} \sum_{\vec{\alpha}_n \vec{\beta}^*_n} \beta^n_j R_{\bar{i}}{}^{\bar{a}\bar{j}}{}_{\bar{c}} D^{\bar{c}} \left(D^{\bar{a}}\right)^{m} \left(D^{\overline{a+1}}\right)^{\beta^n_{a+1}} \cdots \left(D^{\bar{j}}\right)^{\beta^n_j - 1} \cdots \left(D^{\bar{N}}\right)^{\beta^n_N} g. \tag{9}$$

Here, we used

$$[\nabla_i, \nabla_j] \nabla_{k_1} \cdots \nabla_{k_m} f = - \sum_{n=1}^{m} R_{ijk_n}{}^{l} \nabla_{k_1} \cdots \nabla_{k_{n-1}} \nabla_l \nabla_{k_{n+1}} \cdots \nabla_{k_m} f. \tag{10}$$

for $m \geq 1$.

Section 2

One and two dimensional cases

One dimensional case

The Scalar curvature R is defined as $R = g^{i\bar{j}} R_{i\bar{j}} = R_{\bar{l}}{}^{\bar{j}\bar{l}}{}_{\bar{j}}$.

Theorem (Noncommutative Riemannian surfaces)

Let M be a one-dimensional locally symmetric Kähler manifold $(N = 1)$ and f and g be smooth functions on M. The star product with separation of variables for f and g can be described as

$$f * g$$

$$= \sum_{n=0}^{\infty} \left[\left(g^{1\bar{1}} \right)^n \left\{ \prod_{k=1}^{n-1} \frac{2\hbar}{2k + \hbar k (k-1) R} \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial z} \right)^n f \right\} \left\{ \left(g^{1\bar{1}} \frac{\partial}{\partial \bar{z}} \right)^n g \right\} \right.$$

where

$$R = R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}.$$

Two dimensional case (the first term)

Next, we discuss star products on general two-dimensional locally symmetric Kähler manifolds.
According to Proposition **??**, for a two-dimensional locally symmetric Kähler manifold M, $T^1_{\vec{\alpha_1}\vec{\beta_1^*}}$ is given as

$$\begin{pmatrix} T^1_{(1,0),(1,0)} & T^1_{(1,0),(0,1)} \\ T^1_{(0,1),(1,0)} & T^1_{(0,1),(0,1)} \end{pmatrix} = \hbar \begin{pmatrix} g_{1\bar{1}} & g_{1\bar{2}} \\ g_{2\bar{1}} & g_{2\bar{2}} \end{pmatrix}.$$

Next, we estimate $T^2_{\vec{\alpha_2}\vec{\beta_2^*}}$.

Two dimensional case

Proposition

*Let M be a two-dimensional locally symmetric Kähler manifold and f and g be smooth functions on M. $T^2_{\vec{\alpha_2}\vec{\beta_2^*}}$ given in (**??**) is obtained by*

$$\begin{pmatrix} T^2_{(2,0),(2,0)} & T^2_{(2,0),(1,1)} & T^2_{(2,0),(0,2)} \\ T^2_{(1,1),(2,0)} & T^2_{(1,1),(1,1)} & T^2_{(1,1),(0,2)} \\ T^2_{(0,2),(2,0)} & T^2_{(0,2),(1,1)} & T^2_{(0,2),(0,2)} \end{pmatrix}$$
$$= \hbar^2 \begin{pmatrix} (g_{\bar{1}1})^2 & g_{\bar{1}1}g_{\bar{2}1} & (g_{\bar{2}1})^2 \\ 2g_{\bar{1}1}g_{\bar{1}2} & g_{\bar{2}1}g_{\bar{1}2} + g_{\bar{1}1}g_{\bar{2}2} & 2g_{\bar{2}1}g_{\bar{2}2} \\ (g_{\bar{1}2})^2 & g_{\bar{2}1}g_{\bar{2}2} & (g_{\bar{2}2})^2 \end{pmatrix}$$
$$\times \begin{pmatrix} 2 + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}} \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & 1 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}} \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & 2 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}} \end{pmatrix}^{-1}.$$

Section 3

A metric of a complex Grassmann manifold

Complex Grassmann manifold $G_{p,q}$

Definition (Complex Grassmann manifold as a set)

Complex Grassmann manifold $G_{p,q}$ is defined as a set of the whole p dimensional part vector space of $p+q$ dimensional vector space.

$$G_{p,q} := \left\{ V \subset \mathbb{C}^{p+q} : dim_{\mathbb{C}} V = p \right\}$$

In this section, capital letter indices $A, B, C \cdots$ mean $aa', bb', cc' \cdots$. In the inhomogeneous coordinates
$z^I := z^{ii'},\ z^{\bar{I}} := z^{\bar{i}\bar{i}'}, (i = 1, 2, \cdots p, i' = 1, 2, \cdots q)$, the Kähler potential of $G_{p,q}$ is given as

$$\Phi = \ln \left| E_q + Z^\dagger Z \right|, \qquad (11)$$

where $Z = \phi(Y) = (z^I) \in M(q, p; \mathbb{C})$ and $E_q \in M(q, q; \mathbb{C})$ is the unite matrix. From (11), the following facts are derived.

Fubini-Study metric

Fact (Fubini-Study metric)

The Fubini-Study metric $(g_{I\bar{J}})$ *is*

$$ds^2 = 2g_{I\bar{J}}dz^I d\bar{z}^J,$$

then

$$g^{I\bar{J}} = \left(\delta_{ij} + z^{ik'}\bar{z}^{jk'}\right)\left(\delta_{i'j'} + \bar{z}^{kj'}z^{ki'}\right).$$

where

$$g_{I\bar{J}} := g_{ii'\bar{j}\bar{j}'} = \partial_I \partial_{\bar{J}} \Phi = a^{ji} b^{i'j'},\ g^{I\bar{J}} := g^{ii'\bar{j}\bar{j}'} = a_{ij} b_{j'i'}.$$

with

$$a_{ij} = \delta_{ij} + z^{ik'}\bar{z}^{jk'},\ b_{i'j'} = \delta_{i'j'} + \bar{z}^{ki'}z^{kj'}.$$

The Riemannian curvature

Definition

The Riemannian curvature of a Hermitian manifold M is defined as

$$R_{i\bar{j}k}{}^{l} = \partial_i \Gamma^l_{\bar{j}k} - \partial_{\bar{j}} \Gamma^l_{ik} + \Gamma^n_{\bar{j}k} \Gamma^l_{in} - \Gamma^n_{ik} \Gamma^l_{\bar{j}n}$$

where

$$\Gamma^l_{jk} = g^{l\bar{q}} \frac{\partial g_{j\bar{q}}}{\partial z^k}$$

Fact

The Riemannian curvature of a Hermitian manifold M is obtained as

$$[\nabla_i, \nabla_j] \nabla_{k_1} \cdots \nabla_{k_m} f = -\sum_{n=1}^{m} R_{ijk_n}{}^{l} \nabla_{k_1} \cdots \nabla_{k_{n-1}} \nabla_l \nabla_{k_{n+1}} \cdots \nabla_{k_m} f. \quad (12)$$

for $m \geq 1$.

Curvature with Fubini-Study metric

Fact (Curvature with Fubini-Study metric)

The curvature of a complex Grassmann manifold is

$$R_{\bar{A}}{}^{\bar{C}\bar{D}}{}_{\bar{B}} = g^{P\bar{C}} g^{Q\bar{D}} R_{\bar{A}PQ\bar{B}} = -\delta_{\overline{ab'}\,\overline{C}} \delta_{\overline{ba'}\,\overline{D}} - \delta_{\overline{ba'}\,\overline{C}} \delta_{\overline{ab'}\,\overline{D}}, \tag{13}$$

where

$$\delta_{\overline{ab'}\,\overline{cd'}} = \begin{cases} 1 & (a = c, b' = d') \\ 0 & (otherwise) \end{cases} .$$

From these facts, we can derive the recurrence relations to determine star products on the Grassmann manifolds.

Section 4

Deformation quantization for complex Grassmann manifold

Deformation quantization for $\mathbb{C}P^N$

In this subsection, we obtain concrete expression of star products on $\mathbb{C}P^N$. A complex projective space $\mathbb{C}P^N$ is a Grassmann manifold $G_{1,N}$ by definition.

Proposition

*Let M be a complex projective space and f and g be smooth functions on M. The recurrence relation of $T^n_{\vec{\alpha}_n \vec{\beta}^*_n}$ given in (**??**) is*

$$T^n_{\vec{\alpha}_n \vec{\beta}^*_n} = \sum_{d=1}^{N} \frac{\hbar g_{\bar{i}d}}{(1+\hbar-\hbar n)\,\beta^n_i} T^{n-1}_{\vec{\alpha}_n - \vec{e}_d \vec{\beta}^*_n - \vec{e}_i}. \tag{14}$$

Deformation quantization for $\mathbb{C}P^N$

Theorem

Let f and g be smooth functions on a projective space $\mathbb{C}P^N$. A star product with separation of variables on a projective space $\mathbb{C}P^N$ is given as

$$f * g \tag{15}$$

$$= f \cdot g + \sum_{n=1}^{\infty} \sum_{\vec{\alpha_n} \vec{\beta_n^*}} \left| G^{\vec{\alpha_n}, \vec{\beta_n^*}} \right|^{+} \left\{ \prod_{k=0}^{n} \frac{\hbar}{(1 + \hbar - \hbar k)\, \alpha_k^n! \beta_k^n!} \right\} \left(D^{\vec{\alpha_n}} f \right) \left(D^{\vec{\beta_n^*}} g \right).$$

where $|\cdot|^{+}$ is the permanent and $G^{\vec{\alpha_n}, \vec{\beta_n^*}}$ is a matrix made of metrics.

Matrix made of metrics

Definition

A matrix $G^{\vec{\alpha_n},\vec{\beta_n^*}}$ is defined by using the Riemannian metrics on M. Its elements are metrics on M and are located as follows. $\vec{\alpha_n}$ and $\vec{\beta_n}$ are elements of $\mathbb{Z}^N$.

$$G^{\vec{\alpha_n},\vec{\beta_n^*}} = \begin{pmatrix} \tilde{G}_{11} & \cdots & \tilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{n1} & \cdots & \tilde{G}_{nn} \end{pmatrix}$$

where

$$\tilde{G}_{pq} =: g_{p\bar{q}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M\left(\alpha_p^n, \beta_q^n; \mathbb{C}\right)$$

Permanent

A function similar to the determinant is defined on the matrix space.

Definition (Permanent)

Let $C = (C_{k,l})_{1\leq k\leq n, 1\leq l\leq n}$ be a $n \times n$ matrix. We define $|\cdot|^+$ as a $\mathbb{C}$-valued function on $M(n, n; \mathbb{C})$ such that

$$|C|^+ := \sum_{\sigma_n \in S_n} \prod_{k=1}^{n} C_{k,\sigma_n(k)}.$$

This is called "permanent".

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^+ = c_{11}c_{22} + c_{12}c_{21}$$

Deformation quantization for $\mathbb{C}P^N$

Fact

Let f and g be smooth functions on a projective space $\mathbb{C}P^N$. A star product on a projective space $\mathbb{C}P^N$ is given in [28] as

$$
\begin{aligned}
f \tilde{*} g &= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar) g^{\bar{j}_1 k_1} \cdots g^{\bar{j}_n k_n}}{n! \Gamma(1+1/\hbar)} \left(\nabla_{\bar{j}_1} \cdots \nabla_{\bar{j}_n} f \right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g \right) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar)}{n! \Gamma(1+1/\hbar)} \left(D^{k_1} \cdots D^{k_n} f \right) \left(\nabla_{k_1} \cdots \nabla_{k_n} g \right) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(1-n+1/\hbar) g_{\bar{m}_1 k_1} \cdots g_{\bar{m}_n k_n}}{n! \Gamma(1+1/\hbar)} \left(D^{k_1} \cdots D^{k_n} f \right) \left(D^{\bar{m}_1} \cdots D^{\bar{m}_n} g \right) .
\end{aligned} \tag{16}
$$

As mentioned in Section **??**, the star product with separation of variables is uniquely determined. This fact means (15) coincides with (16). This coincidence is easily checked from Definition 11.

Deformation quantization for $G_{2,2}$

In this subsection, we derive the recurrence relation to obtain concrete expression of star products on a Grassmann manifold $G_{2,2}$. The inhomogeneous coordinates are $z^{11'}, z^{12'}, z^{21'}$ and $z^{22'}$. To decide the order of coordinates is useful in order to calculate the finite sum. We set the order:$11' < 12' < 21' < 22'$. In this subsection, j is used as "Not i". That means that if $i = 1$ then $j = 2$ and if $i = 2$ then $j = 1$. For example, if $I = ii' = 11'$, then $ij' = 12', ji' = 21', J = 22'$. If $I = ii' = 12'$, then $ij' = 11', ji' = 22', J = 21'$. A finite sum is defined as

$$\sum_{D=1}^{4} a_D := a_{11'} + a_{12'} + a_{21'} + a_{22'}.$$

Deformation quantization for $G_{2,2}$

Theorem

*Let f and g be smooth functions on $G_{2,2}$. The recurrence relation of $T^{n}_{\vec{\alpha_n}\vec{\beta^*_n}}$ given in (**??**) is*

$$\begin{aligned}
&\beta_I \left(1+\hbar-\hbar\beta^n_I-\hbar\beta^n_{ji'}-\hbar\beta^n_{ij'}\right)\ T^n_{\vec{\alpha_n}\vec{\beta^*_n}} \\
&-\hbar\left(\beta^n_{ij'}+1\right)\left(\beta^n_{ji'}+1\right)\ T^n_{\vec{\alpha_n}\vec{\beta^*_n}-\vec{e_J}+\vec{e_{ij'}}+\vec{e_{ji'}}-\vec{e_I}} \\
&=\hbar g_{\bar{I}I}T^{n-1}_{\vec{\alpha_n}-\vec{e_I}\vec{\beta^*_n}-\vec{e_I}}+\hbar g_{\bar{I}ij'}T^{n-1}_{\vec{\alpha_n}-\vec{e_{ij'}}\vec{\beta^*_n}-\vec{e_I}} \\
&+\hbar g_{\bar{I}ji'}T^{n-1}_{\vec{\alpha_n}-\vec{e_{ji'}}\vec{\beta^*_n}-\vec{e_I}}+\hbar g_{\bar{I}J}T^{n-1}_{\vec{\alpha_n}-\vec{e_J}\vec{\beta^*_n}-\vec{e_I}}.
\end{aligned}$$

for each I.

Star products on a noncommutative $G_{2,2}$ are determined by this formula recursively. For general $G_{p,q}$, the recurrnce relations are determined in a similar way.

Conclusion

Thank you for your attention!
Conclusion

1. The deformation quantization with separation of variables→You can make it by algebraic recurrence relations !
2. Noncommutative Riemannian surfaces
3. Noncommutative Grassmann manifold $\to \mathbb{C}P^n$:not so hard but $\cdots$

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Section

A function similar to the determinant "permanent"

A function similar to the determinant is defined on the matrix space.

Definition (permanent)

Let $C = (C_{k,l})_{1 \le k \le n, 1 \le l \le n}$ be a $n \times n$ matrix. We define $|\cdot|^+$ as a $\mathbb{C}$-valued function on $M(n, n; \mathbb{C})$ such that

$$|C|^+ := \sum_{\sigma_n \in S_n} \prod_{k=1}^{n} C_{k,\sigma_n(k)}.$$

This is called "permanent".

Example

Here we show some examples. These suggest some properties like determinant.

1. $$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^{+} = c_{11}c_{22} + c_{12}c_{21}$$

2. $$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^{+}$$
$$= c_{11}c_{22}c_{33} + c_{11}c_{23}c_{32} + c_{12}c_{21}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} + c_{13}c_{22}c_{31}$$
$$= c_{11}\begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix}^{+} + c_{12}\begin{vmatrix} c_{11} & c_{13} \\ c_{31} & c_{33} \end{vmatrix}^{+} + c_{13}\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^{+}$$

Remark

Similar to a determinant

$$\left|{}^{t}C\right|^{+} = |C|^{+},$$

where ${}^{t}C$ is a transposed matrix of C.

Cofactor expansion of a permanent

Proposition

The following is a proposition similar to cofactor expansion of a determinant.

$$|C|^+ = \begin{vmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nn} \end{vmatrix}^+ = \sum_{j=1}^{n} c_{ij} \begin{vmatrix} c_{11} & \cdots & \hat{c_{1j}} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{c_{i1}} & \cdots & \hat{c_{ij}} & \cdots & \hat{c_{in}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & \hat{c_{nj}} & \cdots & c_{nn} \end{vmatrix}^+$$

Matrix made of metrics

Definition

A matrix $G^{\vec{\alpha_n},\vec{\beta_n^*}}$ is defined by using the Riemannian metrics on M. Its elements are metrics on M and are located as follows. $\vec{\alpha_n}$ and $\vec{\beta_n}$ are elements of $\mathbb{Z}^N$.

$$G^{\vec{\alpha_n},\vec{\beta_n^*}} = \begin{pmatrix} \tilde{G}_{11} & \cdots & \tilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{G}_{n1} & \cdots & \tilde{G}_{nn} \end{pmatrix}$$

where

$$\tilde{G}_{pq} =: g_{p\bar{q}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \in M\left(\alpha_p^n, \beta_q^n; \mathbb{C}\right)$$

Cofactor expansion

From Proposition 0.11, we obtain the following corollary.

Corollary (Cofactor expansion of a permanent)

For a matrix $G^{\vec{\alpha_n},\vec{\beta^*_n}}$,

$$\left| G^{\vec{\alpha_n},\vec{\beta^*_n}} \right|^{+} = \sum_{J=1}^{N} \beta^n_J g_{\bar{J}I} \left| G^{\vec{\alpha_n}-\vec{e_I},\vec{\beta^*_n}-\vec{e_J}} \right|^{+} = \sum_{K=1}^{N} \alpha^n_K g_{\bar{I}K} \left| G^{\vec{\alpha_n}-\vec{e_K},\vec{\beta^*_n}-\vec{e_I}} \right|^{+}.$$

$N = 2$ and $n = 2$

$N=2$ and $n=2$ are substituted in Theorem **??**. The results are listed here. $\vec{\alpha_2}, \vec{\beta_2^*} \in \{(2,0),(1,1),(0,2)\}$ and $i=\{1,2\}$.

$$\hbar^2 (g_{\bar{1}1})^2 = \left(2+\hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\right) T^2_{(2,0),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}}\ T^2_{(2,0),(0,2)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\ T^2_{(2,0),(1,1)}$$

$$\hbar^2 (g_{\bar{1}2})^2 = \left(2+\hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\right) T^2_{(0,2),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}}\ T^2_{(0,2),(0,2)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\ T^2_{(0,2),(1,1)}$$

$$2\hbar^2 g_{\bar{1}1} g_{\bar{1}2} = \left(2+\hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\right) T^2_{(1,1),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}}\ T^2_{(1,1),(0,2)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\ T^2_{(1,1),(1,1)}$$

$$\hbar^2 g_{\bar{1}1} g_{\bar{2}1} = \left(1+\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\right) T^2_{(2,0),(1,1)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\ T^2_{(2,0),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} T^2_{(2,0),(0,2)}$$

$$\hbar^2 g_{\bar{1}2} g_{\bar{2}2} = \left(1+\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\right) T^2_{(0,2),(1,1)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\ T^2_{(0,2),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} T^2_{(0,2),(0,2)}$$

$$\hbar^2 g_{\bar{1}1} g_{\bar{2}2} + \hbar^2 g_{\bar{2}1} g_{\bar{1}2} = \left(1+\hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}}\right) T^2_{(1,1),(1,1)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}}\ T^2_{(1,1),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} T^2_{(1,1),(0,2)}$$

$$\hbar^2 (g_{\bar{2}1})^2 = \left(2+\hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\right) T^2_{(2,0),(0,2)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(2,0),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\ T^2_{(2,0),(1,1)}$$

$$\hbar^2 (g_{\bar{2}2})^2 = \left(2+\hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\right) T^2_{(0,2),(0,2)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(0,2),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\ T^2_{(0,2),(1,1)}$$

$$2\hbar^2 g_{\bar{2}1} g_{\bar{2}2} = \left(2+\hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\right) T^2_{(1,1),(0,2)} + \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(1,1),(2,0)} + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\ T^2_{(1,1),(1,1)}$$

$$\hbar^2 g_{\bar{1}1} g_{\bar{2}1} = \left(1+\hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\right) T^2_{(2,0),(1,1)} + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(2,0),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\ T^2_{(2,0),(0,2)}$$

$$\hbar^2 g_{\bar{2}1} g_{\bar{2}2} = \left(1+\hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\right) T^2_{(0,2),(1,1)} + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(0,2),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\ T^2_{(0,2),(0,2)}$$

$$\hbar^2 g_{\bar{2}1} g_{\bar{1}2} + \hbar^2 g_{\bar{1}1} g_{\bar{2}2} = \left(1+\hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{2}}\right) T^2_{(1,1),(1,1)} + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{2}}\ T^2_{(1,1),(2,0)} + \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{2}}\ T^2_{(1,1),(0,2)}.$$

There are multiple overlapping and tautological equations are omitted.

With (**??**) these are the same as the following equation.

$$
\hbar^2 \begin{pmatrix} (g_{\bar{1}1})^2 & g_{\bar{1}1}g_{\bar{2}1} & (g_{\bar{2}1})^2 \\ 2g_{\bar{1}1}g_{\bar{1}2} & g_{\bar{2}1}g_{\bar{1}2} + g_{\bar{1}1}g_{\bar{2}2} & 2g_{\bar{2}1}g_{\bar{2}2} \\ (g_{\bar{1}2})^2 & g_{\bar{2}1}g_{\bar{2}2} & (g_{\bar{2}2})^2 \end{pmatrix}
$$

$$
= \begin{pmatrix} T^2_{(2,0),(2,0)} & T^2_{(2,0),(1,1)} & T^2_{(2,0),(0,2)} \\ T^2_{(1,1),(2,0)} & T^2_{(1,1),(1,1)} & T^2_{(1,1),(0,2)} \\ T^2_{(0,2),(2,0)} & T^2_{(0,2),(1,1)} & T^2_{(0,2),(0,2)} \end{pmatrix} \left(\begin{matrix} 2 + \hbar R_{\bar{1}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{1}\bar{1}}{}_{\bar{1}} & \hbar \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & 1 + \hbar R_{\bar{2}}{}^{\bar{2}\bar{1}}{}_{\bar{1}} & \hbar \\ \hbar R_{\bar{1}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & \hbar R_{\bar{2}}{}^{\bar{2}\bar{2}}{}_{\bar{1}} & 2 + \end{matrix} \right.
$$

then Proposition 0.7 is proved.

The local coordinate can be defined in a similar way to S. Kobayashi and K. Nomizu pp.160-162[19].

Let U be an open subset of $G_{p,q}$. A chart (U, ϕ) is defined by

$$U := \left\{ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \in M\left(p+q, p; \mathbb{C}\right); |Y_0| \neq 0 \right\}$$

and

$$\phi : U \longrightarrow M\left(q, p; \mathbb{C}\right)$$

where

$$\phi\left(Y\right) = Y_1 Y_0^{-1}.$$

This is a holomorphic map of U onto an open subset of $p \times q$-dimensional complex space.

bibliography I

- A. P. Balachandran, B. P. Dolan, J. -H. Lee, X. Martin and D. O'Connor, "Fuzzy complex projective spaces and their star products," J. Geom. Phys. **43**, 184 (2002) [hep-th/0107099].
- Brian P. Dolan, Oliver Jahn "Fuzzy Complex Grassmannian Spaces and their Star Products," [hep-th/0111020].
- F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 1. Deformations Of Symplectic Structures," Annals Phys. **111** (1978) 61.
  F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory And Quantization. 2. Physical Applications," Annals Phys. **111** (1978) 111.

P. Bieliavsky, S. Detournay and P. Spindel, “The Deformation quantizations of the hyperbolic plane,” Commun. Math. Phys. **289**, 529 (2009) [arXiv:0806.4741 [math-ph]].

M. Bordemann, M. Brischle, C. mmrich, S. Waldmann, “Phase Space Reduction for Star-Products: An Explicit Construction for $\mathbb{C}P^n$,” Lett. Math. Phys. **36** (1996), 357.

M. Cahen, S. Gutt, J. Rawnsley, “ Quantization of Kahler manifolds, II,” Am. Math. Soc. Transl. **337**, 73 (1993).

M. Cahen, S. Gutt, J. Rawnsley, “ Quantization of Kahler manifolds, IV,” Lett. Math. Phys **34**, 159 (1995).

M. De Wilde, P. B. A. Lecomte, “xistence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds,” Lett. Math. Phys. **7**, 487 (1983).

- B. Fedosov, “A simple geometrical construction of deformation quantization,” J. Differential Geom. **40**, 213 (1994).
- M. B. Halima, Tilmann Wurzbacher, “Fuzzy complex Grassmannians and quantization of line bundles” Semin. Univ. Hambg. (2010) 80: 59.
- M. B. Halima, “Construction of certain fuzzy flag manifolds” Rev. Math. Phys. **22**, 533 (2010)
- K. Hayasaka, R. Nakayama and Y. Takaya, “A New noncommutative product on the fuzzy two sphere corresponding to the unitary representation of SU(2) and the Seiberg-Witten map,” Phys. Lett. B **553**, 109 (2003) [hep-th/0209240].
- D. Karabali, V. P. Nair and S. Randjbar-Daemi, “Fuzzy spaces, the M(atrix) model and the quantum Hall effect,” In *Shifman, M. (ed.) et al.: From fields to strings, vol. 1* 831-875 [hep-th/0407007].

- A. V. Karabegov, "On deformation quantization, on a Kahler manifold, associated to Berezin's quantization," Funct. Anal. Appl. **30**, 142 (1996).
- A. V. Karabegov, "Deformation quantizations with separation of variables on a Kahler manifold," Commun. Math. Phys. **180**, 745 (1996) [arXiv:hep-th/9508013].
- A. V. Karabegov, "An explicit formula for a star product with separation of variables," [arXiv:1106.4112 [math.QA]].
- A. V. Karabegov, "PseudoKahler quantization on flag manifolds," dg-ga/9709015.
- Y. Kitazawa, "Matrix models in homogeneous spaces," Nucl. Phys. B **642**, 210 (2002) [hep-th/0207115].

- S. Kobayashi and K. Nomizu, "Foundation of Differential Geometry, volume II," John Wiley and Sons, Inc , 1969
- M. Kontsevich, "Deformation quantization of Poisson manifolds, I," Lett. Math. Phys. **66**, 157 (2003) [arXiv:q-alg/9709040].
- Y. Maeda, A. Sako, T. Suzuki and H. Umetsu, "Deformation Quantization with Separation of Variables and Gauge Theories," Proceedings, 33th Workshop on Geometric Methods in Physics (XXXIII WGMP) : Bialowieza, Poland, June 29-July 5, 2014 ,p.135-144
- C. Moreno, " $*$-products on some Kähler manifolds", Lett. Math. Phys. **11**, 361 (1986).
- C. Moreno, " Invariant star products and representations of compact semisimple Lie groups," Lett. Math. Phys. **12**, 217 (1986).

bibliography VI

- S. Murray and C. Saemann, "Quantization of Flag Manifolds and their Supersymmetric Extensions," Adv. Theor. Math. Phys. **12** (2008) no.3, 641 doi:10.4310/ATMP.2008.v12.n3.a5 [hep-th/0611328].
- T. Ohsaku, "Algebra of noncommutative Riemann surfaces," math-ph/0606057.
- H. Omori, Y. Maeda, and A. Yoshioka, "Weyl manifolds and deformation quantization," Adv. in Math. 85, 224 (1991).
- A. Sako, T. Suzuki and H. Umetsu, "Explicit Formulas for Noncommutative Deformations of CP^N and CH^N," J. Math. Phys. **53**, 073502 (2012) [arXiv:1204.4030 [math-ph]].
- A. Sako, T. Suzuki and H. Umetsu, "Noncommutative CP^N and CH^N and their physics," J. Phys. Conf. Ser. **442**, 012052 (2013).

- A. Sako, T. Suzuki and H. Umetsu, "Gauge theories on noncommutative $\mathbb{C}P^N$ and Bogomol ' nyi-Prasad-Sommerfield-like equations," J. Math. Phys. **56**, no. 11, 113506 (2015). [arXiv:1506.06957hhep-th]]
- A. Sako and H. Umetsu, "Twisted Fock Representations of Noncommutative Kähler Manifolds," arXiv:1605.02600 [math-ph].
- M. Schlichenmaier, "Berezin-Toeplitz quantization and star products for compact Kähler manifolds," Contemp. Math. **583** (2012) 257
- M. Schlichenmaier, "Some naturally defined star products for Kähler manifold," Trav. math. **20** (2012) 187