INDIVIDUAL ERGODIC THEOREMS IN SEMIFINITE VON NEUMANN ALGEBRAS

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Classical Dunford-Schwartz pointwise ergodic theorem

Definition

Let (Ω, μ) be a measure space. A linear operator T on $L^1(\Omega) + L^\infty(\Omega)$ is called a Dunford-Schwartz operator if

$$\|T(f)\|_\infty \leq \|f\|_\infty \ \forall\, f \in L^\infty(\Omega) \ \text{ and } \ \|T(f)\|_1 \leq \|f\|_1 \ \forall\, f \in L^1(\Omega).$$

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Theorem

Let $T : L^1(\Omega) + L^\infty(\Omega) \to L^1(\Omega) + L^\infty(\Omega)$ be a Dunford-Schwartz operator, and let $f \in L^p(\Omega)$, $1 \leq p < \infty$. Then the Cesáro averages

$$A_n(T, f) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(f), \quad n = 1, 2, \ldots$$

converge $\mu-$almost everywhere to some $\widehat{f} \in L^p(\Omega)$.

Overview

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In that paper, it was proved that the Cesáro averages

$$A_n(T, x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \quad n = 1, 2, \ldots \tag{1}$$

generated by a positive Dunford-Schwartz operator T defined on the space $L^1(\mathcal{M}, \tau) + \mathcal{M}$ converge bilaterally almost uniform (b.a.u.) (in Egorov's sence) for every $x \in L^1(\mathcal{M}, \tau)$.

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There were two immediate outstanding problems associated with the result:

(P1) Can b.a.u. convergence $\|e(A_n(T,x) - \widehat{x})e\|_\infty \to 0$, where e is a "big" projection in $\mathcal{M}$, be replaced by generally stronger almost uniform (a.u.) convergence: $\|(A_n(T,x) - \widehat{x})e\|_\infty \to 0$?

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Since then the argument of Junge and Xu has been simplified but no major progress had been attained in answering these questions.

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There is a.u. convergence for all $1 \leq p < \infty$ and if τ is infinite and $\mathcal{M}$ is non-atomic, a.u. convergence of egrodic averages holds for $x \in L^1(\mathcal{M}, \tau) + \mathcal{M}$ if and only if $x \in \mathcal{R}_\mu$, the latter extending far beyond the family of noncommutative L^p-spaces, $1 \leq p < \infty$.

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Besides, we establish a.u. convergence in $\mathcal{R}_\mu$ for a variety of noncommutative individual ergodic theorems, some of which new and some previously known to hold only for b.a.u. convergence.

Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{P}(\mathcal{M})$ be the lattice of projections in $\mathcal{M}$. If $\mathbf{1}$ is the identity of $\mathcal{M}$ and $e \in \mathcal{P}(\mathcal{M})$, we write $e^{\perp} = \mathbf{1} - e$.

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If $1 \leq p < \infty$, then the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ is defined as

$$L^p = L^p(\mathcal{M}, \tau) = \left\{ x \in L^0 : \ \|x\|_p = (\tau(|x|^p))^{1/p} < \infty \right\},$$

where $|x| = (x^*x)^{1/2}$, the absolute value of x. Naturally, $L^{\infty}(\mathcal{M}) = \mathcal{M}$, equipped with the uniform norm $\|\cdot\|_{\infty}$.

Preliminaries

Let $x \in L^0$, and let $\{e_\lambda\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x|$ of x. If $t > 0$, then a non-increasing rearrangement of x is defined as

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A Banach space $(E, \|\cdot\|_E) \subset L^0$ is called symmetric if conditions

$$x \in E, \; y \in L^0, \; \mu_t(y) \leq \mu_t(x) \;\text{ for all }\; t > 0$$

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A Banach space $(E, \|\cdot\|_E) \subset L^0$ is called fully symmetric if

$$x \in E, \ y \in L^0, \ \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt \ \text{ for all } s > 0$$

entail that $y \in E$ and $\|y\|_E \leq \|x\|_E$.

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Theorem

Equipped with the norm

$$\|x\|_{L^1+\mathcal{M}} = \int_0^1 \mu_t(x)dt,$$

$\mathcal{R}_\mu$ is a fully symmetric space.

Proposition

If $\tau(\mathbf{1}) = \infty$, then a symmetric space $E \subset L^1 + \mathcal{M}$ is contained in $\mathcal{R}_\tau$ if and only if $\mathbf{1} \notin E$.

Dunford-Schwartz individual ergodic theorems in $\mathcal{R}_\tau$

A linear operator $T : L^1 + \mathcal{M} \to L^1 + \mathcal{M}$ is called a Dunford-Schwartz operator if

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If a Dunford-Schwartz operator T is positive, we write $T \in DS^+$.

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Given $T \in DS^+$ and $x \in L^1 + \mathcal{M}$, recall that

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$$A_n(T, x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \quad n = 1, 2, \ldots$$

A sequence $\{x_n\} \subset L^0$ is said to converge to $\widehat{x} \in L^0$ almost uniformly (a.u.) (bilaterally almost uniformly (b.a.u.)) if for every $\epsilon > 0$ there exists $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e^\perp) \leq \epsilon$ and $\|(\widehat{x} - x_n)e\|_\infty \to 0$ (respectively, $\|e(\widehat{x} - x_n)e\|_\infty = 0$).

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Theorem (Yeadon 1977)

Let $T \in DS^+$ and $x \in L^1$. Then the averages $A_n(T, x)$ converge b.a.u. to some $\widehat{x} \in L^1$.

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Remark (Chilin-L 2015)

It can be seen that the iterating operators T that were considered by Yeadon can be uniquely extended to a positive Dunford-Schwartz operators, hence the assumption $T \in DS^+$.

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Here is an extension of Yeadon's result:

Theorem (Junge-Xu 2007)

If $T \in DS^+$ and $x \in L^p$, $1 < p < \infty$, then the averages $A_n(T, x)$ converge b.a.u. to some $\widehat{x} \in L^p$. If $p \geq 2$, then these averages converge also a.u.

Dunford-Schwartz individual ergodic theorems in $\mathcal{R}_\tau$

In fact, we have *a.u.* convergence in the above theorems:

Theorem (L 2016)

Let $T \in DS^+$ and $x \in L^p$, $1 \leq p < \infty$. Then the averages $A_n(T, x)$ converge a.u. to some $\widehat{x} \in L^p$.

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Theorem (L 2016)

Let $T \in DS^+$ and $x \in L^p$, $1 \leq p < \infty$. Then the averages $A_n(T, x)$ converge a.u. to some $\widehat{x} \in L^p$.

Proof of this result is based on the following notion.

Definition

Let $(X, \|\cdot\|)$ be a normed space. A sequence of maps $M_n : X \to L^0$ is called bilaterally uniformly equicontinuous in measure (b.u.e.m.) at zero if for every $\epsilon > 0$ and $\delta > 0$ there exists $\gamma > 0$ such that, given $x \in X$ with $\|x\| < \gamma$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying conditions

$$\tau(e^\perp) \leq \epsilon \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \leq \delta.$$

Remark

It is easy to see that, in the commutative case, bilateraly uniform equicontinuity in measure at zero of a sequence $M_n : X \to L^0$ is equivalent to the continuity in measure at zero of the maximal operator $M^*(f) = \sup_n |M_n(f)|, \ f \in X$.

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Proposition (Crucial Step)

Let $(X, \|\cdot\|)$ be a Banach space, $M_n : X \to L^0$ a sequence of linear maps that is b.u.e.m. at zero on X. Then the set

$$\{x \in X : \{M_n(x)\} \text{ converges a.u.}\}$$

is closed in X.

Proposition

The sequence $\{A_n\}$ given by (1) is b.u.e.m. at zero on L^p, $1 \leq p < \infty$.

Since the set $L^p \cap L^2$ is dense in L^p, $1 \leq p < \infty$, and it can be shown that the sequence $\{A_n(x)\}$ converges a.u. whenever $x \in L^2$, the averages $A_n(x)$ converge a.u. for every $x \in L^p$. Q.E.D.

When $X = L^p$, $1 \leq p < \infty$, we have the following result.

Theorem (Noncommutative Banach Principle)

Let $M_n : L^p \to L^0$ be a sequence of positive continuous (with respect to the measure topology in L^0) linear maps such that for every $x \in L^p$ and $\epsilon > 0$ there exists a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying

$$\tau(e^{\perp}) \leq \epsilon \quad \textit{and} \quad \sup_n \|eM_n(x)e\|_{\infty} < \infty.$$

Then the set $\{x \in X : \{M_n(x)\}$ converges a.u.$\}$ is closed in L^p.

Dunford-Schwartz individual ergodic theorems in $\mathcal{R}_\tau$

Here is an extension of the above to $\mathcal{R}_\tau$:

Theorem

Let $T \in DS^+$ and $x \in \mathcal{R}_\tau$. Then the averages $A_n(T,x)$ converge a.u. to some $\widehat{x} \in L^1 + \mathcal{M}$. Moreover, if $E \subset L^1 + \mathcal{M}$ is a fully symmetric space such that $\mathbf{1} \notin E$ ($E \subset \mathcal{R}_\mu$) and $x \in E$, then these averages converge a.u. to some $\widehat{x} \in E$.

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If the algebra $\mathcal{M}$ is non-atomic, then $\mathcal{R}_\mu$ is the largest subspace of $L^1 + \mathcal{M}$ for which we have a.u. convergence of the averages (1):

Theorem

If $x \in (L^1 + \mathcal{M}) \setminus \mathcal{R}_\mu$, then there is $T \in DS^+(\mathcal{M}, \tau)$ such that the sequence $\{A_n(T, x)\}$ does not converge a.u.

Dunford-Schwartz individual ergodic theorems in $\mathcal{R}_\tau$

Let $\{T_{\mathbf{u}} : \mathbf{u} \in \mathbb{R}_+^d\}$ be a semigroup of contractions of L^1 which is continuous in the interior of $\mathbb{R}_+^d$, that is,

$$\|T_{\mathbf{u}}(x) - T_{\mathbf{v}}(x)\|_1 \to 0 \quad \text{as} \quad \mathbf{u} \to \mathbf{v}$$

for all $x \in L^1$ and $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}_+^d$ with $v_i > 0$, $1 \leq i \leq d$.

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Denote

$$A_t(x) = \frac{1}{t^d} \int_{[0,t]^d} T_{\mathbf{u}}(x) d\mathbf{u}, \quad x \in L^1, \ t > 0. \tag{2}$$

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The next theorem is a noncommutative extension of a theorem of Dunford and Schwartz.

Theorem

If $\{T_{\mathbf{u}} : \mathbf{u} \in \mathbb{R}_+^d\} \subset DS^+$ is a semigroup continuous on the interior of $\mathbb{R}_+^d$ and $x \in L^1$. Then the averages $A_t(x)$ given by (2) converge a.u. to some $\widehat{x} \in L^1$.

In particular, we have the following.

Corollary

Let $\{T_s\}_{s\geq 0} \subset DS^+$ be a semigroup that is strongly continuous on L^1 at every $s > 0$. Then the averages

$$\frac{1}{t}\int_0^t T_s(x)ds$$

converge a.u. for every $x \in L^1$ as $t \to \infty$.

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Let $(E, \|\cdot\|_E) \subset L^1 + \mathcal{M}$ be a symmetric space, and let $\{T_{\mathbf{u}} : \mathbf{u} \in \mathbb{R}_+^d\} \subset DS^+$ be a semigroup of contractions in E. We say that $\{T_{\mathbf{u}}\}$ is continuous in the interior of $\mathbb{R}_+^d$ on E, if

$$\|T_{\mathbf{u}}(x) - T_{\mathbf{v}}(x)\|_E \to 0 \text{ as } \mathbf{u} \to \mathbf{v}$$

for all $x \in E$, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}_+^d$ with $v_i > 0$, $1 \leq i \leq d$.

Dunford-Schwartz individual ergodic theorems in $\mathcal{R}_\tau$

Denote, as before,

$$A_t(x) = \frac{1}{t^d} \int_{[0,t]^d} T_{\mathbf{u}}(x) d\mathbf{u}, \quad x \in E, \; t > 0. \tag{3}$$

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Theorem

Let $\{T_{\mathbf{u}} : \mathbf{u} \in \mathbb{R}_+^d\} \subset DS^+$ be a semigroup continuous in the interior of $\mathbb{R}_+^d$ on $\mathcal{R}_\tau$ and L^1. Then for every $x \in \mathcal{R}_\tau$ the averages (3) converge a.u. as $t \to \infty$ to some $\widehat{x} \in \mathcal{R}_\tau$.

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Theorem

Let $\mathbf{1} \notin E \subset L^1 + \mathcal{M}$ be a fully symmetric space, and let $\{T_{\mathbf{u}} : \mathbf{u} \in \mathbb{R}_+^d\} \subset DS^+$ be a semigroup continuous in the interior of $\mathbb{R}_+^d$ on $\mathcal{R}_\tau$, L^1, and E. Then for every $x \in E$, the averages (3) converge a.u. as $t \to \infty$ to some $\widehat{x} \in E$.

Weighted noncommutative individual ergodic theorems

Let $\mathbb{C}_1$ be the unit circle in $\mathbb{C}$. A function $P : \mathbb{Z} \to \mathbb{C}$ is said to be a trigonometric polynomial if $P(k) = \sum_{j=1}^{s} z_j \lambda_j^k$, $k \in \mathbb{Z}$, for some $s \in \mathbb{N}$, $\{z_j\}_1^s \subset \mathbb{C}$, and $\{\lambda_j\}_1^s \subset \mathbb{C}_1$.

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A sequence $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C}$ is called bounded Besicovitch if

(a) $\sup_k |\beta_k| \leq C < \infty$;

(b) for every $\epsilon > 0$ there exists a trigonometric polynomial P such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |\beta_k - P(k)| < \epsilon.$$

Weighted noncommutative individual ergodic theorems

Theorem (Chilin-L-Skalski 2005)

Assume that $\mathcal{M}$ has a separable predual. Let $T \in DS^+$, and let $\{\beta_k\}$ be a bounded Besicovitch sequence. Then the averages

$$B_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \beta_k T^k(x) \tag{4}$$

converge b.a.u. for every $x \in L^1$ to some $\widehat{x} \in L^1$.

Here is an extension of the previous theorem:

Theorem

Let $\mathcal{M}$, T, and $\{\beta_k\}$ be as above. Then for every $x \in \mathcal{R}_\tau$ the averages (4) converge a.u. to some $\widehat{x} \in L^1 + \mathcal{M}$.

Weighted noncommutative individual ergodic theorems

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$$A_n(x,\lambda) = \frac{1}{n}\sum_{k=0}^{n-1}\lambda^k T^k(x), \tag{5}$$

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Definition

We say that $x \in L^1 + \mathcal{M}$ satisfies Wiener-Wintner property and we write $x \in WW$ if, given $\epsilon > 0$, there exists a projection $e \in P(\mathcal{M})$ with $\tau(e^\perp) \leq \epsilon$ such that the sequence $\{A_n(x, \lambda)e\}$ converges in $(\mathcal{M}, \|\cdot\|_\infty)$ for all $\lambda \in \mathbb{C}_1$.

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Theorem

$L^1 \subset WW$, that is, for every $x \in L^1$ and $\epsilon > 0$ there exists a projection $e \in P(\mathcal{M})$ such that $\tau(e^\perp) \le \epsilon$ and $\{A_n(x,\lambda)e\}$ converges in $(\mathcal{M}, \|\cdot\|_\infty)$ for all $\lambda \in \mathbb{C}_1$.

Applications

Every individual ergodic theorem for $\mathcal{R}_\tau$ above (except possibly the previous one) is valid for any noncommutative fully symmetric space $E \subset \mathcal{R}_\tau$ (with the limit $\widehat{x} \in E$). We shall give a few examples of noncommutative fully symmetric subspaces of $\mathcal{R}_\tau$.

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Recall that $\tau(\mathbf{1}) = \infty$. As we have noticed, a symmetric space $E \subset L^1 + \mathcal{M}$ is contained in $\mathcal{R}_\tau$ if and only if $\mathbf{1} \notin E$.

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1. Let Φ be an Orlicz function, that is, $\Phi : [0, \infty) \to [0, \infty)$ is a convex continuous at 0 function such that $\Phi(0) = 0$ and $\Phi(u) > 0$ if $u \neq 0$.

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Denote

$$L^\Phi = \left\{ x \in L^0 : \ \tau\left(\Phi\left(\frac{|x|}{a}\right)\right) < \infty \ \text{ for some } \ a > 0 \right\}.$$

be the corresponding noncommutative Orlicz space.

Applications

Let

$$\|x\|_\Phi = \inf\left\{a > 0 : \tau\left(\Phi\left(\frac{|x|}{a}\right)\right) \le 1\right\}$$

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Since $\tau(\mathbf{1}) = \infty$, we have $\tau\left(\Phi\left(\frac{1}{a}\right)\right) = \infty$ for all $a > 0$, hence $\mathbf{1} \notin L^\Phi$, hence $L^\Phi \subset \mathcal{R}_\mu$.

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2. A space $(E, \|\cdot\|_E)$ is said to have *order continuous norm* if $\|x_n\|_E \downarrow 0$ whenever $x_n \in E_+$ and $x_n \downarrow 0$.

If $E = E(\mathcal{M}, \tau)$ is a noncommutative fully symmetric space with order continuous norm, then $\tau(\{|x| > \lambda\}) < \infty$ for all $x \in E$ and $\lambda > 0$, so $E \subset \mathcal{R}_\tau$.

Applications

If $E = E(0, \infty)$ is a symmetric function space, then the space

$$E(\mathcal{M}) = \{x \in L^0 : \mu_t(x) \in E\} \text{ with } \|x\|_{E(\mathcal{M})} = \|\mu_t(x)\|_E$$

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Since $\Lambda_\varphi(0, \infty)$ is a fully symmetric function space and $\Lambda_\varphi(0, \infty) \subset \mathcal{R}_\mu(0, \infty)$ whenever $\varphi(\infty) = \infty$, the noncommutative fully symmetric space $\Lambda_\varphi(\mathcal{M}, \tau)$ is contained in $\mathcal{R}_\tau$.

Applications

4. Let $E(0,\infty)$ be a fully symmetric function space, and let $D_s : E(0,\infty) \to E(0,\infty)$, $s > 0$, be the bounded linear operator given by $D_s(f)(t) = f(t/s),\ \ t > 0$.

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THANK YOU!