Special Symmetric Spaces

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Introduction

Projective Euclidean and symmetric spaces play important role in the differential geometry. It is well-known [9] that space with an affine connection is called projective Euclidean, if every geodesic of this space is mapped onto straight line in Euclidean spaces $\bar{E}_n$. Symmetric spaces are characterized by covariantly constant curvature tensor. These spaces started to study P. A. Shirokov [1, 2] and then were studied by many other authors, especially É. Cartan [3] and A. Lichnerowicz [6]. The geodesic mappings and transformations of symmetric spaces studied G. Takemo, M. Ikeda [13], N. S. Sinyukov [12], J. Mikeš, I. Hinterleitner [5, 4, 7, 8].

Theorem

In non flat symmetric projective Euclidean spaces A_n exists a projective coordinate system $x \equiv (x^1, x^2, \dots, x^n)$ in which the components of an affine connection has following form

$$\Gamma_{ij}^h = \delta_i^h \psi_j + \delta_j^h \psi_i \tag{1}$$

where δ_i^h is the Kronecker delta,

$$\begin{gathered} \psi = -1/2 \ \ln|\varphi| \\ \varphi = e_1(x^1)^2 + e_2(x^2)^2 + \dots + e_k(x^k)^2 + 1 \\ e_\tau = \pm 1, \ \tau = 1, 2, \dots, k, \ 1 \le k \le n. \end{gathered} \tag{2}$$

This result specifies result by P. A. Shirokov [1, 2], which give us information, that the set of those spaces depends on $(n+1)(n+2)/2$ real parameters. Continuation with this problem, we have following theorem with stronger property.
Besides of the previous, the observer or a user of an automaton can formulate

Theorem

Symmetric projective Euclidean spaces are clearly identified by signature of symmetric covariantly constant tensor field.

Projective Euclidean spaces

Let A_n be a space with affine connection ∇. In A_n, we define the torsion curvature and the Ricci tensors:

$$S_{ij}^{h} = \Gamma_{ij}^{h} - \Gamma_{ji}^{h} \tag{3}$$

$$R_{ij} = -R_{ij\alpha}^{\alpha} \tag{4}$$

where $\Gamma_{ij}^{h}(x)$ are components of ∇.

It is also defined the Weyl tensor of projective curvature

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}) + \\ + \frac{1}{n+1}\left(\delta^h_i R_{[jk]} + \frac{1}{n+1}(\delta^h_k R_{[ij]} - \delta^h_j R_{[ik]})\right) \quad (5)$$

where $[jk]$ denotes an alternation of indices j and k.
Space A_n with symmetric Ricci tensor (i.e. $R_{ij} = R_{ji}$) is called equiaffine. In the equiaffine space is the Weyl tensor simplified to

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right). \quad (6)$$

Space A_n is called flat (or affine), if there exists an affine coordinate system x for which $\Gamma^h_{ij}(x) = 0$. It is known, that the tensor criterion for these spaces is that the curvature and torsion tensor are vanished. In natural way, we can implement Euclidean and pseudo-Euclidean metrics, thus we call them Euclidean E_n.

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* if any geodesic curve in A_n is mapped onto geodesic curve in $\bar{A}_n$. In a common coordinate system x respective f, the necessary and sufficient condition of geodesic mapping $f\colon A_n \to \bar{A}_n$ is

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{7}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of ∇ and $\bar{\nabla}$, $\psi_i(x)$ are components of a linear form.

For curvature, Ricci and Weyl projective tensor in A_n and $\bar{A}_n$ the following formulas hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_i \psi_{[kj]} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} \tag{8}$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\ \psi_{ij} + \psi_{[ij]}. \tag{9}$$

$$\bar{W}^h_{ijk} = W^h_{ijk}. \tag{10}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$ and " , " it denotes covariant derivative. Space A_n ($n > 2$) is a projective Euclidean if and only if $W^h_{ijk} = 0$. It is known, that A_n ($n > 2$) is projective Euclidean if and only if the curvature and Ricci tensor has following form:

$$R^h_{ij} = \delta^h_i \psi_{[jk]} + \delta^j_j \psi_{ik} - \delta^h_k \psi_{ij}, \tag{11}$$

This tensor necessarily satisfies the conditions

$$\psi_{ij,k} = \psi_{ik,j}. \tag{12}$$

The Ricci tensor of projective Euclidean space has form

$$R_{ij} = (n-1)\ \psi_{ij} - \psi_{[ij]}. \tag{13}$$

where ψ_{ij} is a tensor.
For this tensor following equation holds:

$$\psi_{ij,k} = \psi_{ik,j}. \tag{14}$$

From this follows that projective Euclidean space is equiaffine if and only if

$$\psi_{ij} = \psi_{ji}.$$

Symmetric projective Euclidean spaces

Since 1925 P. A. Shirokov studied symmetric projective Euclidean space. Symmetric space A_n is characterized by covariantly constant curvature tensor:

$$R^h_{ijk,l} = 0. \tag{15}$$

Name of that space comes from É. Cartan, who studied them more precisely [3]. To symmetric space are devoted many papers, for example [7, 8, 10, 11, 12].

P. A. Shirokov [1, 2] proved, that in non-flat symmetric projective Euclidean space, there exists projective coordinate system x in which the components of affine connection ∇ have form

$$\Gamma^h_{ij} = \delta^h_i \psi_j + \delta^h_j \psi_i \tag{16}$$

$$\varphi \equiv a_{\alpha\beta} x^\alpha x^\beta + b_\alpha x^\alpha + c \tag{17}$$

where $\psi = -1/2\ \ln|\varphi|$, $a_{ij},\ b_i,\ c$ are constants, $a_{ij} = a_{ji} \not\equiv 0$.
From this result it follows that set of symmetric projective Euclidean spaces depends on $(n+1)(n+2)/2$ real parameters i.e. $a_{ij} (= a_{ji})$, b_i and c.
Next, we are proving the first one Theorem.

Proof.

Let A_n be a symmetric projective Euclidean space. On the base of results by P.A.Shirokov [1], the components of ∇ have the form (14) and (13). Now, we will show that the space A_n with (14) and (13) we can affinelly map onto $\bar{A}_n$ with (1) and (2). So, we have to prove that between A_n and $\bar{A}_n$ exists locally affine mapping, see [8, p. 12]. Because the spaces A_n and $\bar{A}_n$ are equiaffine then their curvature tensors have following form :

$$\begin{aligned} R^h_{ijk} &= \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}; \\ \bar{R}^h_{ijk} &= \delta^h_k \bar{\psi}_{ij} - \delta^h_j \bar{\psi}_{ik}; \\ \bar{\psi}_{ij} &\equiv \bar{\psi}_{i,j} - \bar{\psi}_i \bar{\psi}_j; \\ \psi_i \equiv \psi_{,i} \quad &; \quad \psi = -\ln\sqrt{\varphi}; \\ \bar{\psi}_i \equiv \bar{\psi}_{,i} \quad &; \quad \bar{\psi} = -\ln\sqrt{\bar{\varphi}}; \end{aligned} \tag{18}$$

where $\psi \equiv \psi_{i,j} - \psi_i \psi_j$. □

Function φ and $\bar{\varphi}$ are determined by formula (17) and (2) respective. More precisely:

$$\varphi \equiv a_{\alpha\beta}x^{\alpha}x^{\beta} + b_{\alpha}x^{\alpha} + c \qquad (19)$$

$$\bar{\varphi} = e_1(\bar{x}^1)^2 + e_2(\bar{x}^2)^2 + \cdots + e_k(\bar{x}^k)^2 + 1 \qquad (20)$$

Let x_0 be a point in A_n and $U \subset A_n$ is a coordinate neighborhood of x_0 and let x_0 is in the center of that coordinate. We construct locally affine mapping $f : A_n \to \bar{A}_n$, i.e. $f : \bar{x}^h = \bar{x}^h(x)$. We also suppose that mapping fulfills

$$\bar{x}^h(x_0) = 0.$$

For functions $\bar{x}^h(x)$, which are realizing affine mapping between A_n and $\bar{A}_n$ the following equations are fullfiled (see [8, p. 12]):

$$\begin{gathered} \partial_i \bar{x}^h = \bar{x}^h_i \\ \partial_j \bar{x}^h_i = \Gamma^h_{ij}(x)\bar{x}^h_{\alpha} - \bar{\Gamma}^h_{\alpha\beta}(\bar{x}(x))\bar{x}^{\alpha}_i x^{\beta}_j , \end{gathered} \qquad (21)$$

where $|\bar{x}^h_i| \neq 0$.

The integrabilitiy conditions of system (20) have the form:

$$\bar{x}^h(x_0) = 0; \qquad \bar{x}_i^h(x_0) = \overset{0}{\bar{x}}{}_i^h. \tag{22}$$

Next, we add (18) in to integrability conditions (22) and we have

$$\delta_k^h \psi_{ij} - \delta_j^h \psi_{ik} = 0, \tag{23}$$

where $\psi_{ij} = \psi_{ij} - \bar{\psi}_{\alpha\beta} \bar{x}_i^\alpha \bar{x}_j^\beta$. From (23) follows that ψ_{ij} is vanishing, i.e.

$$\psi_{ij} = \bar{\psi}_{\alpha\beta} \bar{x}_i^\alpha \bar{x}_j^\beta. \tag{24}$$

Because $\psi_{ij,k} = 0$ and $\bar{\psi}_{ij|k} = 0$ holds, the differential prolongation of (24) are identically fulfilled.

The end!

Thank you for your attention!

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