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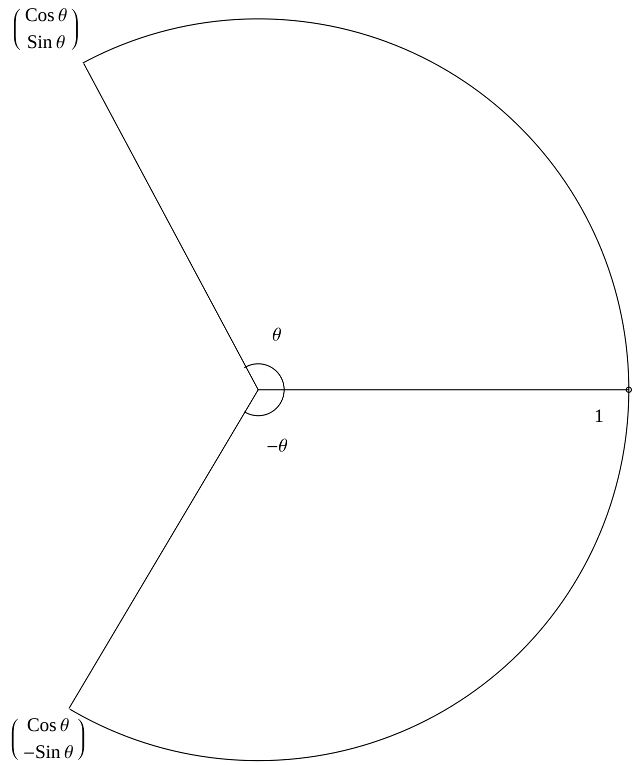

A circular arc

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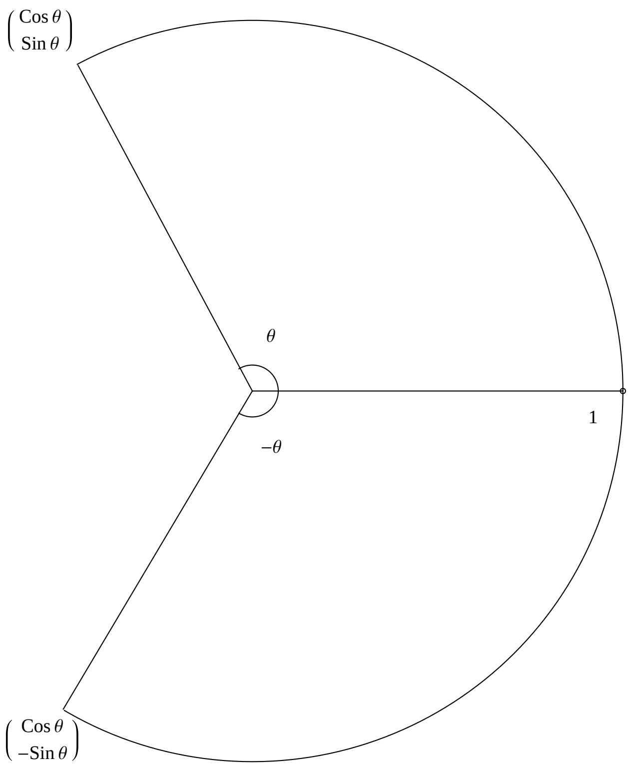

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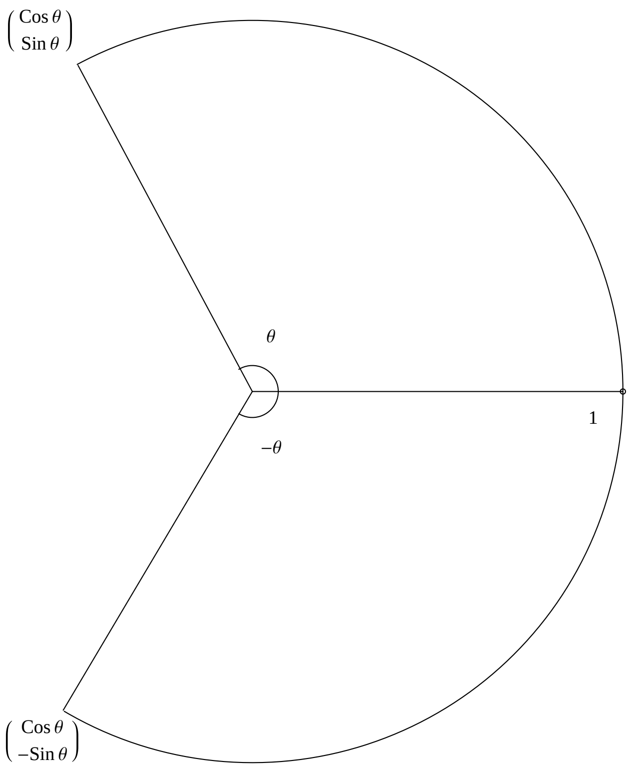
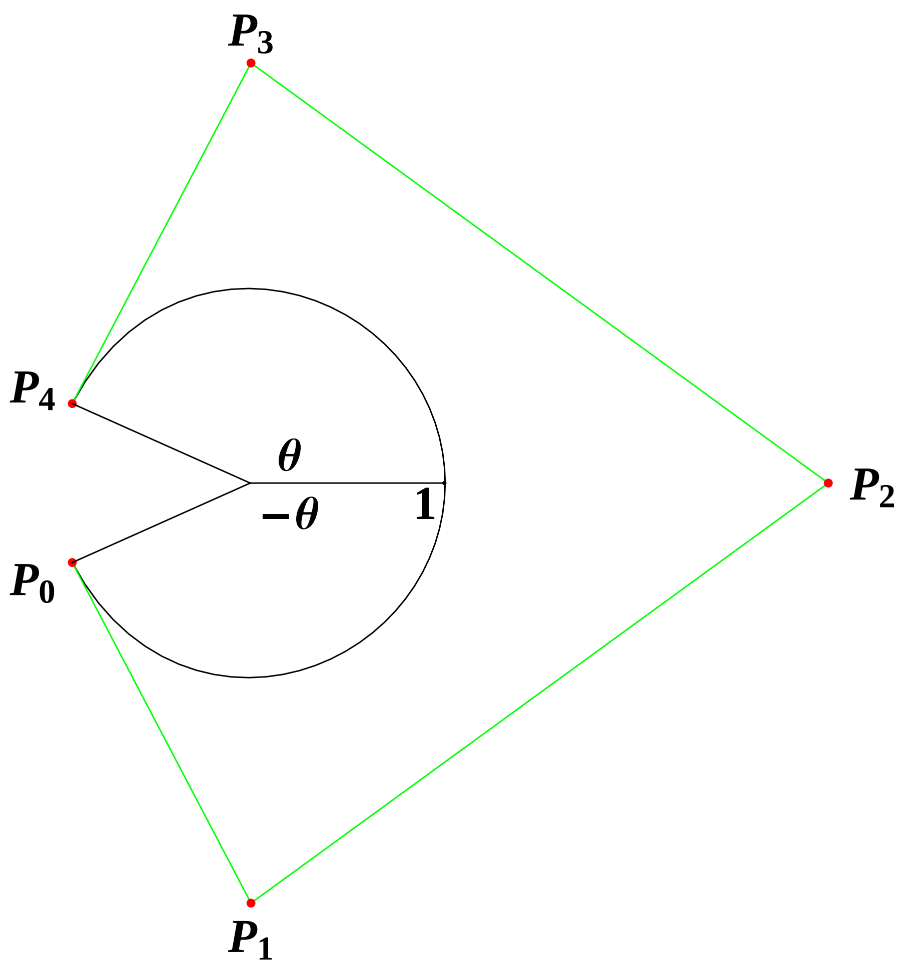
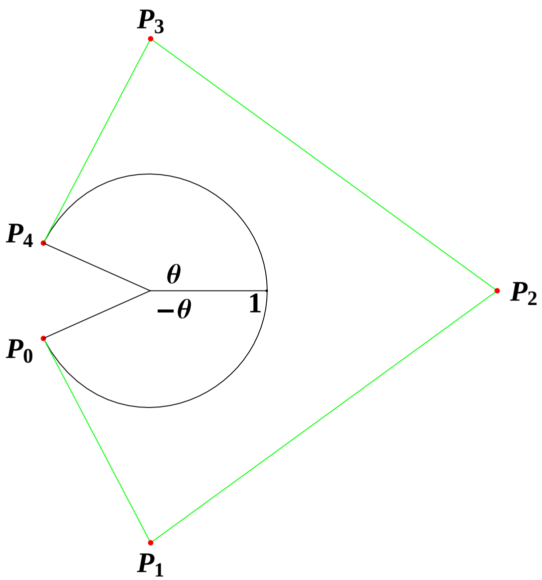


A circular arc

- Minimum Error: p minimizes $\max_{t\in[0,1]} |e(t)|$
- $e(t)$ equioscillates 9 times
- p approximates c with order 8



Quartic Bézier curve



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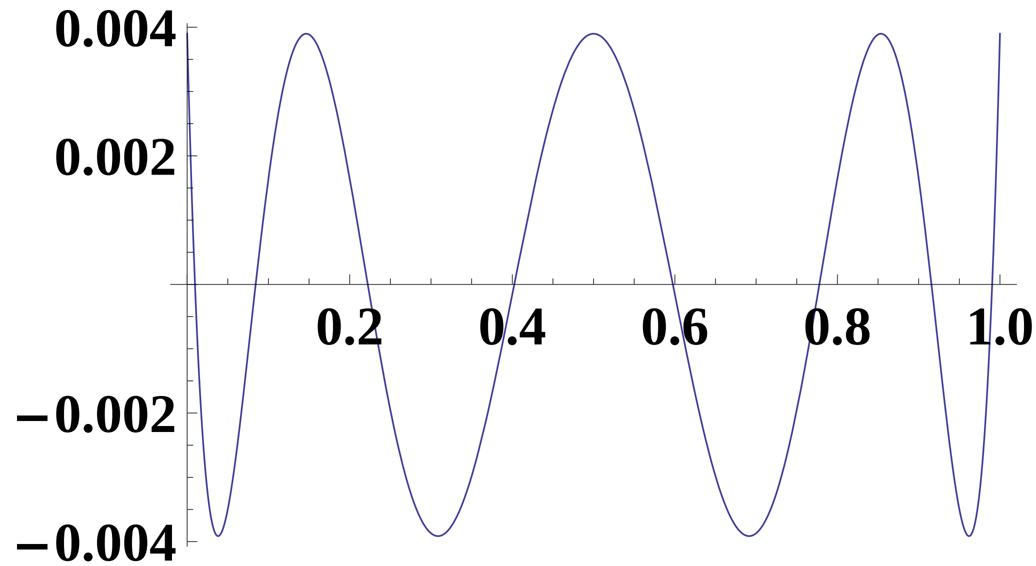

Euclidean Error

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The Problem

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The square root limits the possibility of further progress. Thus, to avoid radicals, the squares of the components of the parametrization to the circle are used.

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So, the Euclidean error $E(t)$ is replaced by the following error function

$$e(t) := x^2(t) + y^2(t) - 1.$$

Bézier Curve

Bézier Curve

BÉZIER FORM: $p(t)$ of degree 4 is given in Bézier form:

$$p(t) = \sum_{i=0}^{4} p_i B_i^4(t) =: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad 0 \leq t \leq 1,$$

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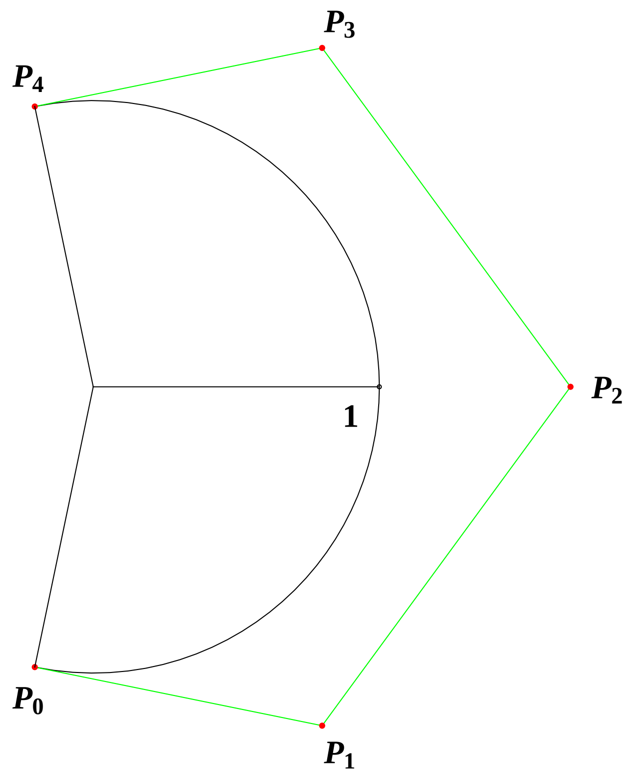
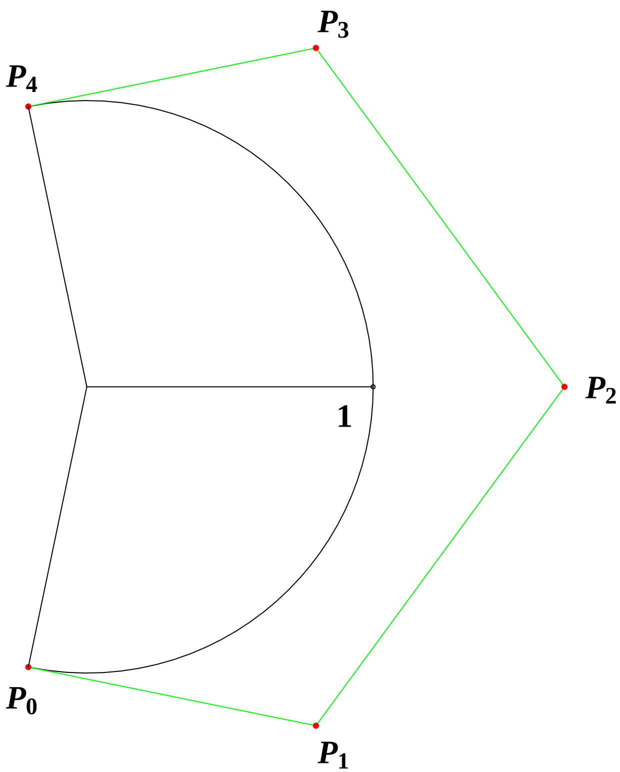
$B_0^4(t) = (1-t)^4$, $B_1^4(t) = 4t(1-t)^3$, $B_2^4(t) = 6t^2(1-t)^2$, $B_3^4(t) = 4t^3(1-t)$ and $B_4^4(t) = t^4$: Bernstein polynomial degree 4.

Possible Bézier Curve

Possible Bézier Curve



Possible Bézier points of circular arc

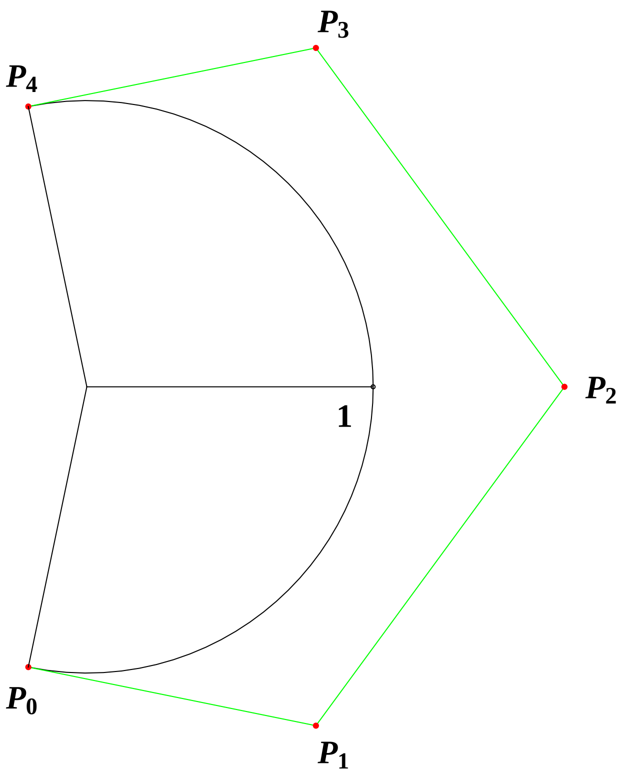


Bézier points of circular arc

The Bézier points:

$$p_0 = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}, \quad p_1 = \begin{pmatrix} \gamma \\ -\zeta \end{pmatrix}, \quad p_2 = \begin{pmatrix} \xi \\ 0 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} \gamma \\ \zeta \end{pmatrix}, \quad p_4 = \begin{pmatrix} -\alpha \\ \beta \end{pmatrix}.$$



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The Bézier curve:

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\alpha \left(B_0^4(t) + B_4^4(t)\right) + \gamma(B_1^4(t) + B_3^4(t)) + \xi B_2^4(t) \\ \beta \left(B_4^4(t) - B_0^4(t)\right) + \zeta \left(B_3^4(t) - B_1^4(t)\right) \end{pmatrix},$$

Free Parameters

There are 5 free parameters

$$\alpha, \beta, \gamma, \zeta, \xi$$

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And solving the resulting equation using a computer algebra system.

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used to: have the polynomial curve p comply with the conditions of the approximation problem by substituting $x(t)$ and $y(t)$ into $e(t)$ and solving the resulting equation using a computer algebra system.

Thereafter, it is shown that these values satisfy the approximation conditions.

Theorem 1:

The Bézier curve with the Bézier points, wherein

$$\alpha = \alpha^* := 0.9165842681395256, \qquad \beta = \beta^* := 0.4094945413544973,$$
$$\gamma = \gamma^* := 0.003898650263063 2704, \qquad \zeta = \zeta^* := 2.164585487675063,$$
$$\xi = \xi^* := 2.977392956397 2596$$

fulfils the following three conditions:

- p minimizes the infinity norm of the error function $\max_{t\in[0,1]} |e(t)|$
- p approximates c with order 8,
- the error function $e(t)$ equioscillates 9 times in $[0, 1]$.

The error functions satisfy:

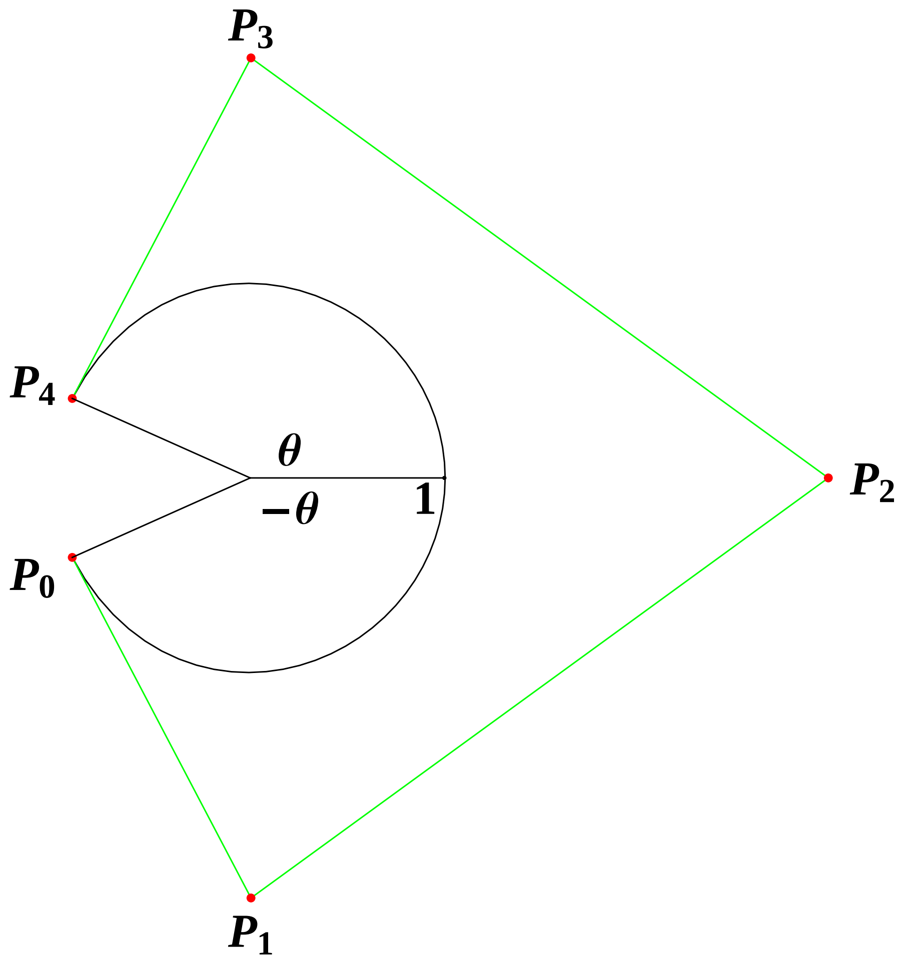
$$-\frac{1}{2^7} \leq e(t) \leq \frac{1}{2^7},$$

$$-\frac{1}{2^7(2-\epsilon)} \leq E(t) \leq \frac{1}{2^7(2+\epsilon)},$$

where

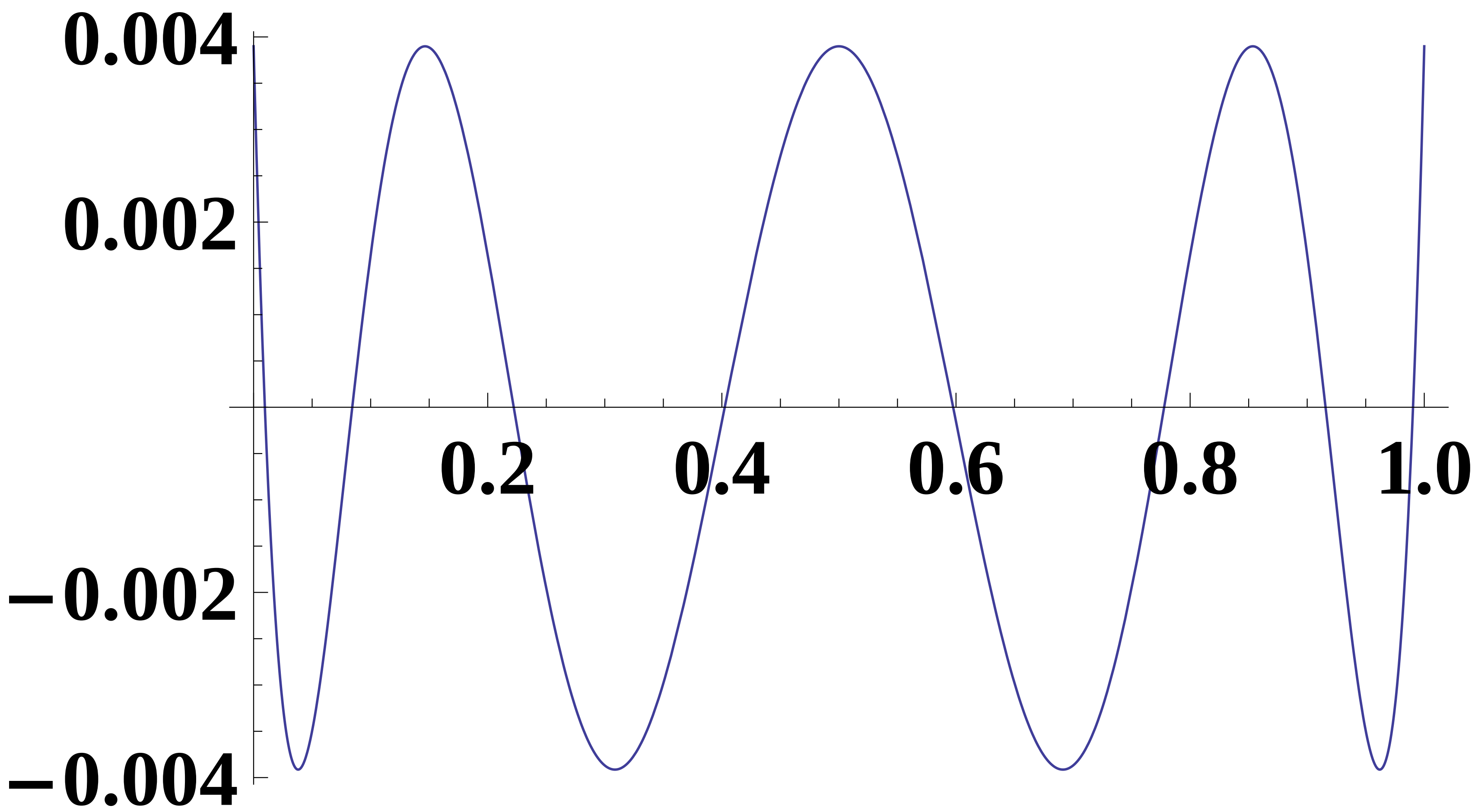
$$\epsilon = \max_{0 \leq t \leq 1} |E(t)| \approx 2^{-8}, \forall t \in [0,1].$$

Quartic Bézier Curve



Quartic Bézier curve in Theorem 1.

The Error



Euclidean Error of the quartic Bézier curve in Theorem 1.

PROPERTIES of the quartic Bézier curve:

Proposition I: The zeros of $e(t)$ and $E(t)$:

$$t_1 = \frac{1}{2}(1 + \cos(\frac{\pi}{16})) = 0.990393, \qquad t_2 = \frac{1}{2}(1 + \cos(\frac{3\pi}{16})) = 0.915735$$

$$t_3 = \frac{1}{2}(1 + \sin(\frac{3\pi}{16})) = 0.777785, \qquad t_4 = \frac{1}{2}(1 + \sin(\frac{\pi}{16})) = 0.597545,$$

$$t_5 = \frac{1}{2}(1 - \sin(\frac{\pi}{16})) = 0.402455, \qquad t_6 = \frac{1}{2}(1 - \sin(\frac{3\pi}{16})) = 0.222215,$$

$$t_7 = \frac{1}{2}(1 - \cos(\frac{3\pi}{16})) = 0.0842652, \qquad t_8 = \frac{1}{2}(1 - \cos(\frac{\pi}{16})) = 0.00960736.$$

These roots also satisfy

$$t_i + t_j = 1, \quad \text{for} \quad i + j = 9.$$

Proposition II: The extreme values of e(t) and $E(t)$ occur at

$$\tilde{t}_0 = 1, \quad \tilde{t}_1 = \frac{1}{2}(1 + \cos(\frac{\pi}{8})) = 0.96194, \quad \tilde{t}_2 = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) = 0.853553,$$

$$\tilde{t}_3 = \frac{1}{2}(1+\sin(\frac{\pi}{8})) = 0.691342, \quad \tilde{t}_4 = \frac{1}{2}, , \quad \tilde{t}_5 = \frac{1}{2}(1-\sin(\frac{\pi}{8})) = 0.308658.$$

$$\tilde{t}_6 = \frac{1}{2}(1-\frac{1}{\sqrt{2}}) = 0.146447, \quad \tilde{t}_7 = \frac{1}{2}(1-\cos(\frac{\pi}{8})) = 0.0380602, \quad \tilde{t}_8 = 0.$$

These parameters satisfy the equality:

$$\tilde{t}_i + \tilde{t}_j = 1, \quad \text{for} \quad i + j = 8.$$

Proposition III: the values of $e(t)$ and $E(t)$ at $\tilde{t}_i$'s are given by:

$$e(\tilde{t}_{2i}) = \frac{1}{128},\ i = 0, \ldots, 4,$$

$$e(\tilde{t}_{2i+1}) = \frac{-1}{128},\ i = 0, \ldots, 3.$$

$$E(\tilde{t}_{2i}) = 3.9 \times 10^{-3},\ i = 0, \ldots, 4,$$

$$E(\tilde{t}_{2i+1}) = -3.9 \times 10^{-3},\ i = 0, \ldots, 3.$$

Therefore,

$$\frac{-1}{128} \leq e(t) \leq \frac{1}{128},$$

$$-3.9 \times 10^{-3} \leq E(t) \leq 3.9 \times 10^{-3},\ t \in [0, 1].$$

Proposition IV: For every $t \in [0, 1]$, the errors of approximating the circular arc using the quartic Bézier curves in Theorem 1 are given by:

$$e(t) = 256t^8 - 1024t^7 + 1664t^6 - 1408t^5 + 660t^4 - 168t^3 + 21t^2 - t + \frac{1}{128}.$$

Examples and Comparisons

Most of existing schemes are for cubic Bézier curves.

Bézier (1986): interpolate end points, point in middle, 3×10^{-4}.

Blinn (1987): used values and tangents at end points, 4×10^{-4}.

De Boor, Höllig, and Sabin (1988): values of positions, tangents, and curvatures at endpoints and got approximation order 6.

Rababah (1992): get the high order approximation of $2n$, for a polynomial of degree n. Therein, a circular arc is represented as an example using data at one or 2 points, 2×10^{-3}.

Dokken, Dæhlen, Lyche, and Mørken (1990): a scheme using geometric properties of circle, 1×10^{-4}.

Goldapp (1991): presented different types of cubic approximations of circular arcs of order 6, 2×10^{-4}.

However, some schemes use quartic Bézier curves.

Ahn and Kim (1997) described quartic scheme that has the parameters 0 and 1 of multiplicity 4 as roots of the error function with error 4×10^{-5}.

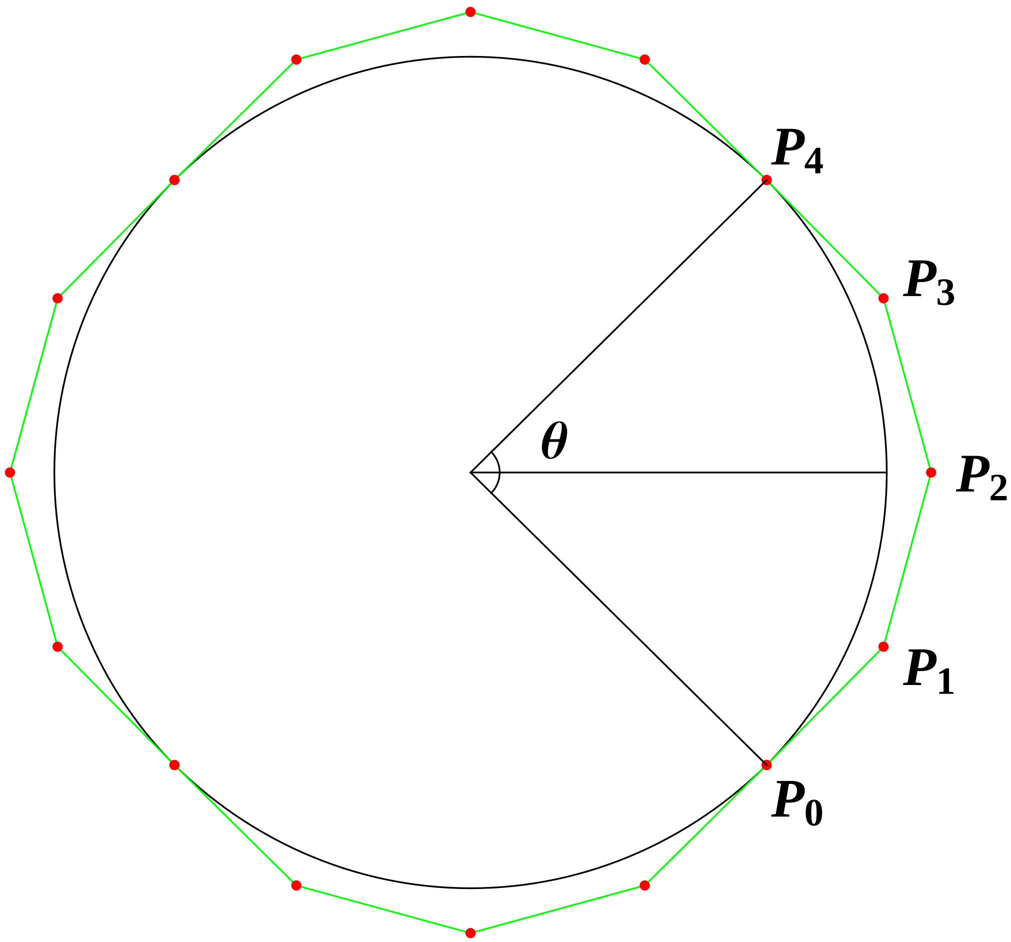
Ahn, Kim, and Shin (2004) described other scheme has the parameters $0, 0.5, 1$ with multiplicities $3, 2, 3$, respectively as roots of the error function with error 4×10^{-6}.

Kim and Ahn (2007, 2013) presented a scheme that is curvature-continuous with error $7.6 \times 10^{-6}, \quad 2 \times 10^{-6}$.

The last approximation is the best result so far.

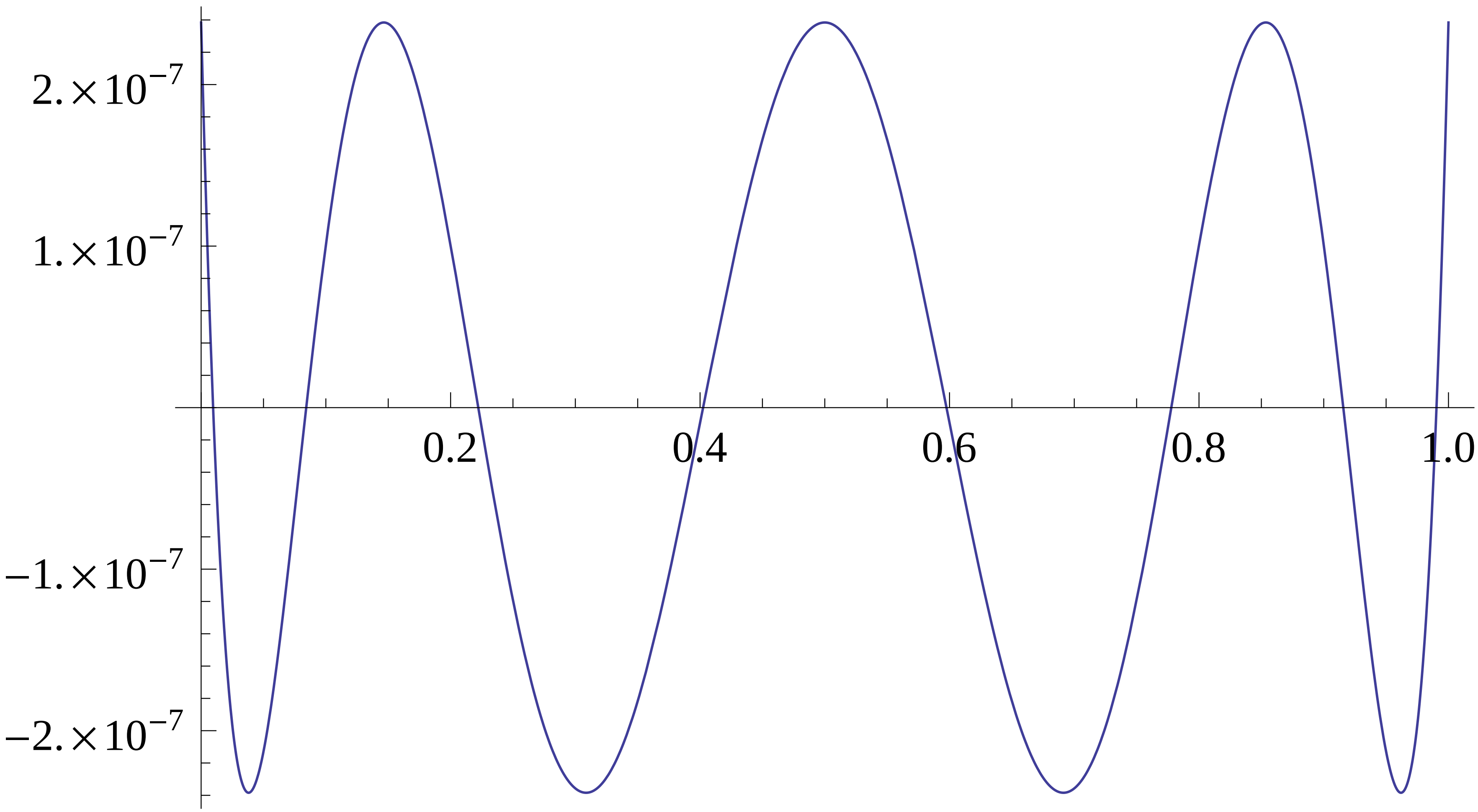
In our scheme, The error function has 8 distinct roots that have Chebyshev distribution in the interval $[0, 1]$.
The quartic Bézier curve has the least uniform deviation from the x-axis with maximum error of 2×10^{-7}.

4 Quartic Bézier curves



4 quarters of quartic approximating Bézier curves.

The Error



Euclidean error of one quarter

Conclusions

The best uniform approximation of a circular arc with parametrically defined polynomial curves of degree 4 is explicitly given.

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The maximum error 2×10^{-7} and thus outperforming the approximations given so far in the literature.

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The error function equioscillates 9 times;

the approximation order is 8.

The approximation intersects the circular arc 8 times with maximum error 2×10^{-7} and thus outperforming the approximations given so far in the literature.

Numerical examples are given to demonstrate the efficiency and simplicity of the approximation method. The method in this paper is C^0-continuous by construction. There are methods in the literature that are G^1- and G^2-continuous.

Open Problems

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1. study quartic approximation with G^k–continuity, $k = 1, 2$, using equioscillating error functions and constrained Chebyshev polynomials.

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4. (Suggested by Paul Sablonniére) It would be interesting to compare our curve with the quartic exponential Euler spline defined by Schoenberg and studied by de Boor.

Thank you!

Questions?

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