On bifurcation of closed geodesics

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Introduction

The theory of geodesics began by Bernoulli and Euler, detailed description of this problem can be found in [1, 2, 8], [6, pp. 88-91]. Geodesics are extremals of length on surfaces and also on Riemannian and pseudo-Riemannian manifolds. Lagrange obtained new properties of geodesics on surfaces, e.g. he proved that ideal rubber (spring) on surface has a position of the geodesic.
It is well-know that there exist one and the only one geodesic going through given point in given direction. This statement is valid for surfaces where Christoffel symbols are differentiable. The proof follows from analysis of ordinary differential equations.
On the other hand, if the Christoffel symbols are continuous, then geodesics exist for above mentioned.

Geodesics I

Let (M, ∇) be a manifold M with affine connection ∇. In local chart (U, x) the connection ∇ is defined with its components $\Gamma^h_{ij}(x)$.
The curve $\gamma(t)$ on M is called *geodesic* if its tangent vector $\gamma'(t)$ is recurrent along it, see [3, 4, 6, 5, 7, 9].
Locally, on the curve γ exists *canonical* parameter s for which satisfy:

$$\nabla_s \dot{\gamma} = 0. \qquad (1)$$

In this case vector field $\dot{\gamma}(s)$ is parallel along γ.
In chart (U, x) equation (1) has following form:

$$\ddot{x}^h(s) + \Gamma^h_{ij}\big(x(s)\big)\, \dot{x}^i(s)\, \dot{x}^j(s) = 0 \qquad (2)$$

where $x^h = x^h(s)$ are equations of geodesic γ on chart (U, x).
Often, we call geodesic those curves that satisfy equations (2).

Geodesics II

These equations (2) can be written in the following form of ordinary differential equations of first order, respective unknown functions $x^h(s)$ and $\lambda^h(s)$:

$$\begin{aligned} \dot{x}^h(s) &= \lambda^h(s) \\ \dot{\lambda}^h(s) &= -\Gamma^h_{ij}\big(x(s)\big)\,\dot{x}^i(s)\,\dot{x}^j(s). \end{aligned} \tag{3}$$

Here $\lambda^h(s)$ is a tangent vector of $\gamma(s)$ at the point $x^h(s)$. Initial conditions of equations (3)

$$x^h(s) = x^h_0, \quad \text{and} \quad \lambda^h(s) = \lambda^h_0 \tag{4}$$

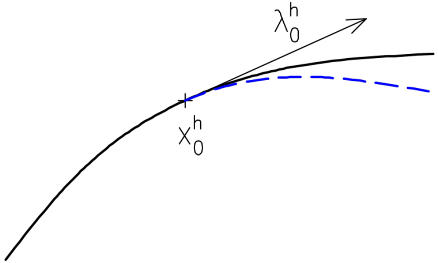
satisfy, that geodesic goes through point x^h_0 in direction λ^h_0 $(\neq 0)$.

Geodesics III

From general theory of ordinary differential equations follows that equations (3) with initial conditions (4) have solution if $\Gamma^h_{ij}(x)$ are continuous functions.
If functions $\Gamma^h_{ij}(x)$ are differentiable, then solution of Cauchy problem (3) and (4) is *unique*.
Moreover, last mentioned applies even if functions $\Gamma^h_{ij}(x)$ satisfy Lipschitz condition.

Geodesic Bifurcation

Bifurcation of geodesic is studied in [10].
Here, we define geodesic bifurcation as situation when different geodesics go through one point and have *same* tangent vector.



We show bifurcation of geodesics on surfaces of revolution, where two different geodesics go through the same point in the *same* direction.

Geodesic on Surface of Revolution

Let $\mathcal{S}_2$ be a surface of revolution given by the equations:

$$x = r(u)\cos v, \quad y = r(u)\sin v, \quad z = z(u) \tag{5}$$

where v is parameter from $(-\pi, \pi)$ and $u \in I \subset \mathbb{R}$, where $I = \langle u_1, u_2 \rangle$.
In these equations we exclude meridian corresponding to $v = \pi$. Naturally, we also exclude "poles"which correspond to $r(u) = 0$.
Rotational surface $\mathcal{S}$ given by equations (5) has following metric

$$\mathrm{d}s^2 = \left(r'^2(u) + z'^2(u)\right)\mathrm{d}u^2 + r'^2(u)\,\mathrm{d}v^2.$$

Let us choose parameter u as a lenght parameter of forming curve $(r(u), 0, z(u))$ then $r'^2(u) + z'^2(u) = 1$. In this case, metric of surface $\mathcal{S}$ is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + r^2(u)\,\mathrm{d}v^2.$$

Let $f(u) \equiv r^2(u)$. The equations (2) of geodesics on surface $\mathcal{S}$ can be written in following form:

$$\ddot{u} = \frac{1}{2} f'(u) \dot{v}^2 \tag{6a}$$

$$\ddot{v} = -\frac{f'(u)}{f(u)} \dot{u} \dot{v}. \tag{6b}$$

Because s is parameter of lenght, tangent vector of these geodesics is unitary, i.e. first integral applies:

$$\dot{u}^2 + f(u) \dot{v}^2 = 1. \tag{7}$$

Trivially, we verify that u-coordinate curves ($u = s$, $v = \mathrm{const}$, i.e. "meridian") are geodesic. In general, same does not apply for the v-coordinates, v-curves are geodesic if and only if $f'(u) = 0$ (they are also called *gorge circles*).

Further, let us study geodesics, which are none of mentioned above. Suppose that $v(s) \neq 0$, i.e. $\dot{v}(s) \neq 0$. Then we can rewrite equation (6b) in form

$$\frac{\ddot{v}}{\dot{v}} = -\frac{f'(u)}{f(u)}\,\dot{u}.$$

After modification and integration by s we get:

$$\dot{v} = \frac{C_1}{f(u)}, \quad C_1 \in \mathbb{R} \tag{8}$$

By using (8), from (7) we get $\dot{u}^2 + f(u)\,\frac{{C_1}^2}{f^2(u)} = 1$, therefore:

$$\dot{u} = \sqrt{1 - \frac{{C_1}^2}{f(u)}}. \tag{9}$$

Finally the equations (8) and (9) determine system of differential equations of first order.

Example of Geodesic Bifurcation

Now we construct example of rotational surface $\mathcal{S}$, where above mentioned bifurcation exists.
Let us choose functions

$$r(u) = \frac{1}{\sqrt{1 - u^{2\alpha}}} \quad \left(\Rightarrow f(u) = \frac{1}{1 - u^{2\alpha}} \right), \quad u \in (-1, 1).$$

Function r has to be differentiable so Christoffel symbols exist and equations of geodesics can be written. On the other hand Christoffel symbols can not satisfy Lipschitz condition and, of course, can not be differentiable (there would be an unique solution and bifurcation would not exist).

Theorem

On above mentioned surface of revolution $\mathcal{S}$ exist geodesic bifurcations for $\alpha \in (0, 1)$.

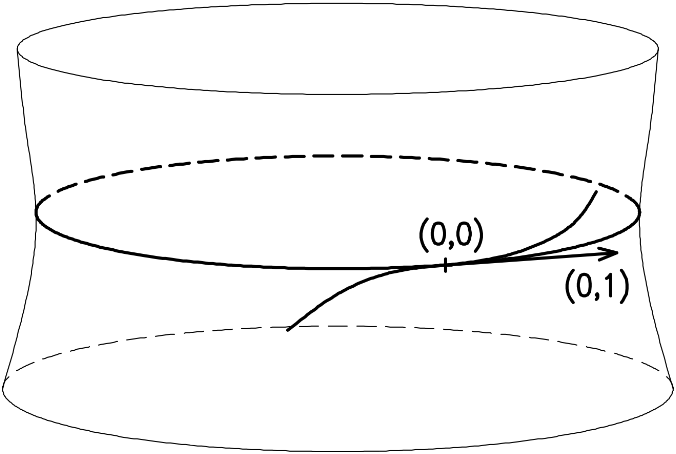
Proof.

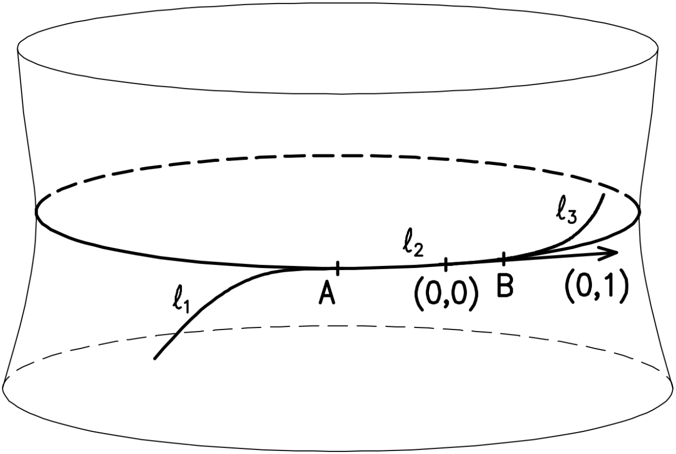
The statement can be proved by existence of geodesics given by the equations:

$$\begin{aligned} I.&\qquad u = 0, \quad v = s \\ II.&\qquad u = \left((1-\alpha)\, s\right)^{\frac{1}{1-\alpha}}, \quad v = s - \frac{\left((1-\alpha)\, s\right)^{\frac{1+\alpha}{1-\alpha}}}{1+\alpha}. \end{aligned} \tag{10}$$

We can verify that curves given by the equations (10) are geodesics by direct substitution to fundamental equations (8) and (9)

These two geodesics go through same point $(0, 0)$ and have same tangent vector $(0, 1)$. The consequence is that through this point in this direction goes infinite number of geodesics and the gorge circle (mentioned above) is one of them. □


(0,0)
(0,1)



ℓ_3
ℓ_2
ℓ_1
A
(0,0)
B
(0,1)

Thank you for attention.

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