Systems with a Position-Dependent-Mass and Symmetry-Preserving Inverse Problems in Lagrangian Dynamics

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CLASSICAL INVERSE PROBLEMS

► (Flat) Bertrand theorem: If $\mathbf{F} = -\frac{dU}{dt}$ central and "bounded orbit" implies "closed orbit" $\Rightarrow U = kr^{-1}$ or $U = kr^2$

► First example of large hierarchy of different inverse problems.[1]

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[1]Bertrand J 1852 *J. Math. Pures Appl.* **17** 121; Bertrand J 1877 *Comp. Rend.* **84** 671; Dainelli U 1880 *Giorn. Mat.* **18** 271; Imshenetsky V G 1882 *Mém. Soc. Sci. Bordeaux* **4** 31; von Helmholtz H 1887 *J. reine angew. Math.* **100** 137; Suslov G K. On a force function admitting given integrals. Kiev 1890; Ermakov V P 1891 *Recueil Math.* **15** 611; Darboux G 1901 *Arch. Neerl.* **6** 371; Dall'Acqua F A 1908 *Math. Annalen* **66** 398; Kasner E 1909 *Differential Geometric Aspects of Dynamics* (AMS, New York); Eisenhart LP 1909 *A Treatise on the Differential Geometry of Curves and Surfaces* (Dover, New York); Burgatti P 1911 *Rend. Acad. Lincei* **20** 108; Schaefer C 1919 *Die Prinzipe de Dynamik* (W. de Gruyter, Berlin); Charlier C L 1927 *Die Mechanik des Himmels* (Veit, Leipzig); Douglas D 1941 *Trans. Amer. Math. Soc.* **50** 71; Whittaker E T 1944 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Dover Pub., New York); Chaplygin S A. Sobr. soch. t. 1, GITTL 1948; Erugin N P 1952 *PMM* **16** 59; Drâmbã C 1963 *St. Cerc. Astron. Acad. Rep. Pop. Rom* **8** 7; Steudel H 1967 *Ann. Physik* **475** 11; Szebehely V G 1967 *Theory of Orbits* (Academic Press, New York); Galiullin A S 1972 *Differenc. uravneniya* **8** 1535; Djukič Dj S, Vujanovič B D 1975 *Acta Mech.* **23** 17; Rosenberg R 1977 *Analytical Dynamics of Discrete Systems* (Plenum Press, New York); Santilli R M 1978 *Foundations of Theoretical Mechanics vol. I* (Springer, New York); Szebehely V, Broucke R 1981 *Celestial Mechanics* **30** 395; Галиуллин А С 1981 *Обратные задачи динамики, Наука, Москва*; Mertens R 1981 *ZAMM* **61** T252; Kolokol'tsev V 1983 Izv. Akad. Nauk SSSR **46** 994; Stavinschi M 1985 *Topics Astrophys. Astron. Sp. Sci.* **1** 41; Melis A, Borghero F 1986 *Meccanica* **21** 71; Bozis G, Ichtiaroglou S 1987 *Inverse Problems* **3** 213; Shorokhov S G 1988 *Celestial Mechanics* **44** 193; Kozlov VV, Harin A O 1992 *Celestial Mechanics* **54** 393; Bozis G 1995 *Inverse Problems* **11** 687; Mei F X 2000 *Acta Mech.* **141** 135; Ziglin S L 2001 *Doklady Phys.* **46** 570; Borisov AV, Mamaev IS 2004 *Reg. Chaotic Dyn.* **9** 265; Voyatzi D, Ichtiaroglou S 2005 *Celestial Mechanics* **93** 331; Kotoulos T, Ichtiaroglou S 2006 *J. Geom. Phys.* **56** 2447; Mušicki, Dj. 2012 *Acta Mech.* **223** 2117; RCS 2014 *Comm. Nonlinear Sci. Num. Simulat.* **19** 2602; Casetta L, Pesce C P 2015 *Acta Mech.* **226** 63; RCS 2016 *Acta Mech.* **227** 1941.

Dainelli and Zhukovski problems

► Given plane curves $\varphi(x,y)=\alpha$, find the most general forces such that φ is an admissible trajectory.

► Components:[2]

$$F_x = u(x,y)\left(\varphi_{xy}\varphi_y - \varphi_{yy}\varphi_x\right) + \frac{1}{2}\varphi_y\left(\varphi_y\frac{\partial u}{\partial x} - \varphi_x\frac{\partial u}{\partial y}\right), \qquad (1)$$

$$F_y = u(x,y)\left(\varphi_{xy}\varphi_x - \varphi_{xx}\varphi_y\right) + \frac{1}{2}\varphi_x\left(\varphi_x\frac{\partial u}{\partial y} - \varphi_y\frac{\partial u}{\partial x}\right). \qquad (2)$$

► No constraint on $u(x,y)$. When $\mathbf{F} = -\frac{\partial V}{\partial x}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} - \frac{\partial V}{\partial y}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}$?

► Explicitly:

$$\frac{1}{2}\varphi_x\varphi_y\left(u_{xx}-u_{yy}\right) + \varphi_y\left(u\varphi_{xy^2} + u_x\varphi_{y^2} + \frac{3}{2}u_x\varphi_{x^2} + \frac{1}{2}u_y\varphi_{xy} + u\varphi_{x^3}\right)$$
$$-\varphi_x\left(u\varphi_{y^3} + \frac{1}{2}u_x\varphi_{xy} + \frac{3}{2}u_y\varphi_{y^2} + u_y\varphi_{x^2} + u\varphi_{yx^2}\right)\frac{1}{2}u_{xy}\left(\varphi_y^2 - \varphi_x^2\right) = 0. \qquad (3)$$

[2]Normalized units ($m=1$) and $u(x,y)=k(x,y)^2$.

- Extrapolation to manifolds (surfaces in $\mathbb{R}^3$) due to Zhukovsky
- Free motion on surface S with line element $\overline{ds^2}$

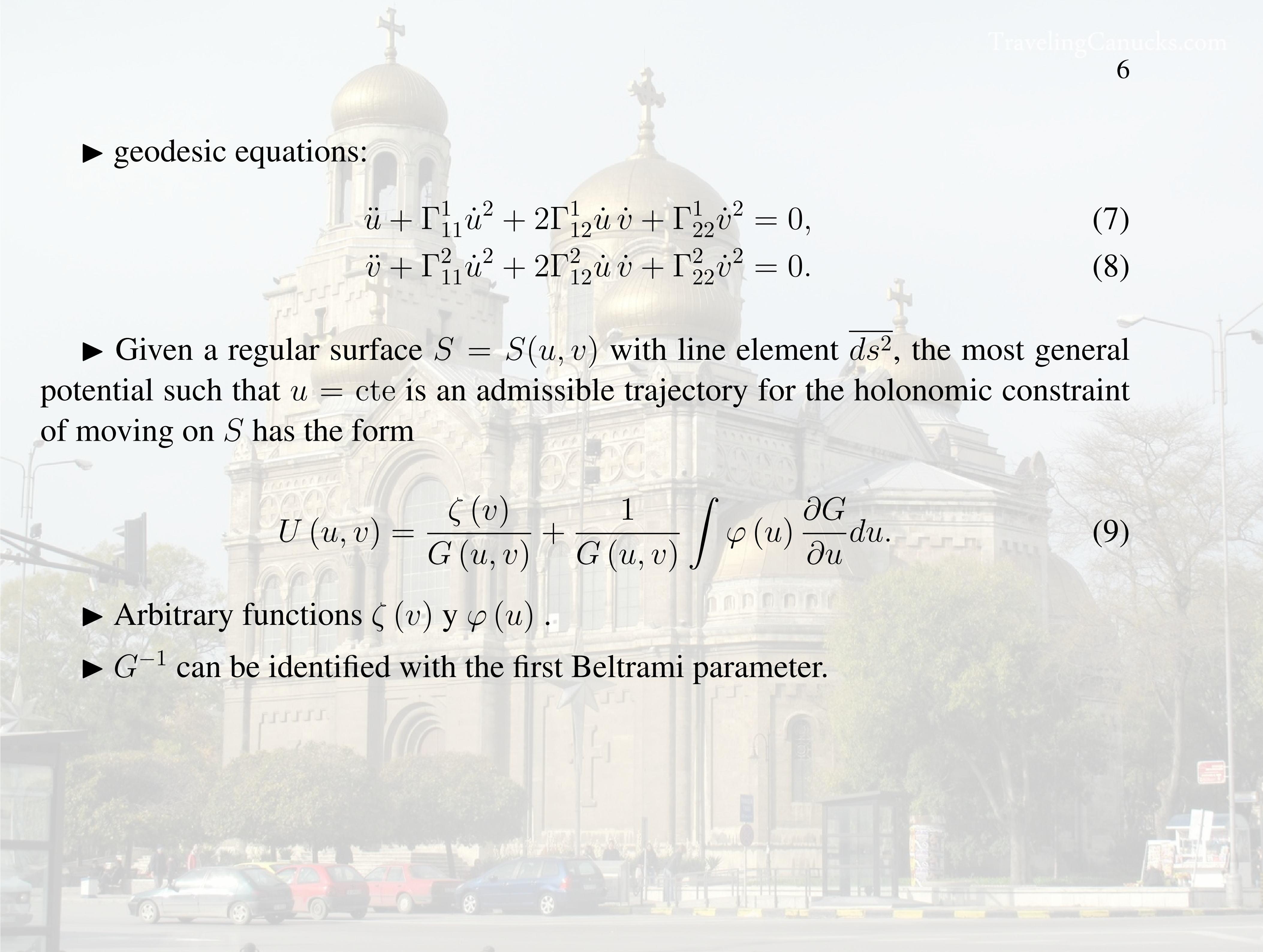
$$L = T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\left(E\,\dot{u}^2 + 2F\,\dot{u}\,\dot{v} + G\,\dot{v}^2\right) \tag{4}$$

- Type II Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \left(\frac{\partial L}{\partial q^i}\right) = 0,\;\; i = 1, 2;\;\; (q^1, q^2) = (u, v) \tag{5}$$

- Using the associated linear connection

$$\begin{aligned}
\Gamma^1_{11} &= \tfrac{1}{2\Delta}\left(G\tfrac{\partial E}{\partial u} + F\tfrac{\partial E}{\partial v} - 2F\tfrac{\partial F}{\partial u}\right), & \Gamma^1_{12} &= \tfrac{1}{2\Delta}\left(G\tfrac{\partial E}{\partial v} - F\tfrac{\partial G}{\partial u}\right), \\
\Gamma^1_{22} &= \tfrac{1}{2\Delta}\left(2G\tfrac{\partial F}{\partial v} - G\tfrac{\partial G}{\partial u} - F\tfrac{\partial G}{\partial v}\right), & \Gamma^2_{11} &= \tfrac{1}{2\Delta}\left(2E\tfrac{\partial F}{\partial u} - F\tfrac{\partial E}{\partial u} - E\tfrac{\partial E}{\partial v}\right), \\
\Gamma^2_{12} &= \tfrac{1}{2\Delta}\left(E\tfrac{\partial G}{\partial u} - G\tfrac{\partial E}{\partial v}\right), & \Gamma^2_{22} &= \tfrac{1}{2\Delta}\left(F\tfrac{\partial G}{\partial u} + E\tfrac{\partial G}{\partial v} - 2F\tfrac{\partial F}{\partial v}\right).
\end{aligned} \tag{6}$$



▶ geodesic equations:

$$\ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\,\dot{v} + \Gamma^1_{22}\dot{v}^2 = 0, \tag{7}$$

$$\ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\,\dot{v} + \Gamma^2_{22}\dot{v}^2 = 0. \tag{8}$$

▶ Given a regular surface $S = S(u,v)$ with line element $\overline{ds^2}$, the most general potential such that $u = \mathrm{cte}$ is an admissible trajectory for the holonomic constraint of moving on S has the form

$$U(u,v) = \frac{\zeta(v)}{G(u,v)} + \frac{1}{G(u,v)} \int \varphi(u) \frac{\partial G}{\partial u} du. \tag{9}$$

▶ Arbitrary functions $\zeta(v)$ y $\varphi(u)$.

▶ G^{-1} can be identified with the first Beltrami parameter.

The Szebehely-Mertens equation

▶ Motivation: Determine potentials for which an observed trajectory $\varphi(u,v)=C$ is contained in a given surface S. [3]

▶ u,v not necessarily ortogonal: $g=\overline{ds^2}=Edu^2+2Fdudv+Gdv^2$

▶ From $\frac{d}{dt}\varphi(u,v)=\varphi_u\dot{u}+\varphi_v\dot{v}=0$ we get

$$\dot{u}=H(u,v)\left(F\varphi_u+G\varphi_v\right),\quad \dot{v}=-H(u,v)\left(E\varphi_u+F\varphi_v\right) \tag{10}$$

▶ Components

$$\ddot{u}=(H_u\dot{u}+H_v\dot{v})\left(F\varphi_u+G\varphi_v\right)+H(u,v)\left(\varphi_u(F_u\dot{u}+F_v\dot{v})+F(\varphi_u^2\dot{u}+\varphi_{uv}\dot{v})\right)+ \\ H(u,v)\left(\varphi_v(G_u\dot{u}+G_v\dot{v})+G(\varphi_{uv}\dot{u}+\varphi_{v^2}\dot{v})\right) \tag{11}$$

$$-\ddot{v}=(H_u\dot{u}+H_v\dot{v})\left(E\varphi_u+F\varphi_v\right)+H(u,v)\left(\varphi_u(E_u\dot{u}+E_v\dot{v})+E(\varphi_u^2\dot{u}+\varphi_{uv}\dot{v})\right)+ \\ H(u,v)\left(\varphi_v(F_u\dot{u}+F_v\dot{v})+F(\varphi_{uv}\dot{u}+\varphi_{v^2}\dot{v})\right) \tag{12}$$

▶ Assumption: There exist $E_0\left(\varphi(u,v)\right)$ y $U(u,v)$ satisfying

$$U(u,v)=\frac{1}{2}g(\mathbf{w},\mathbf{w})-E_0(\varphi(u,v));\quad \mathbf{w}=(\dot{u},\dot{v}) \tag{13}$$

[3] Considered by Szebehely for the planar case (angular momentum conservation in satellite systems), generalization due, among others, to Mertens R 1981 *ZAMM* **61** T252. See also Shorokhov S G 1998 *Celestial Mechanics* **44** 193.

- As $E_0 = \text{cons}$ on each orbit $\varphi(u,v) = \alpha \Longrightarrow \frac{\partial E_0}{\partial u} = \frac{\partial E_0}{\partial \varphi}\frac{\partial \varphi}{\partial u}$ y $\frac{\partial E_0}{\partial v} = \frac{\partial E_0}{\partial \varphi}\frac{\partial \varphi}{\partial v}$.
- Now, from (13) we deduce that

$$\frac{\partial U}{\partial u} = \frac{1}{2}\frac{\partial}{\partial u}\left(g(\mathbf{w},\mathbf{w})\right) - \frac{\partial E_0}{\partial \varphi}\frac{\partial \varphi}{\partial u} = 0; \; \frac{\partial U}{\partial v} = \frac{1}{2}\frac{\partial}{\partial v}\left(g(\mathbf{w},\mathbf{w})\right) - \frac{\partial E_0}{\partial \varphi}\frac{\partial \varphi}{\partial v} = 0.$$

- Developing this expression, taking into account (10), (11) y (12)[4]

$$\left(G\frac{\partial \varphi}{\partial u} - F\frac{\partial \varphi}{\partial v}\right)\frac{\partial U}{\partial u} + \left(E\frac{\partial \varphi}{\partial v} - F\frac{\partial \varphi}{\partial u}\right)\frac{\partial U}{\partial v} = 2W\left(E_0 - U\right) \tag{14}$$

- where

$$W = \frac{1}{A}\left(\Delta\left(\varphi_v^2\varphi_{u^2} - 2\varphi_u\varphi_v\varphi_{uv} + \varphi_u^2\varphi_{v^2}\right) - B_1\left(G\varphi_u - F\varphi_v\right) - B_2\left(E\varphi_v - F\varphi_u\right)\right)$$

- and $A = E\varphi_v^2 - 2F\varphi_u\varphi_v + G\varphi_v^2$, $B_1 = \frac{1}{2}E_u\varphi_v^2 - E_v\varphi_u\varphi_v + \left(F_v - \frac{1}{2}G_u\right)\varphi_u^2$,
- $B_2 = \frac{1}{2}G_v\varphi_u^2 - G_u\varphi_u\varphi_v + \left(F_u - \frac{1}{2}E_v\right)\varphi_v^2$.

[4] Paraphrasing Eliè Cartan: *“Il se trouve que c’est un calcul extrêmement pénible!”*.

SPECIAL CASE: $S = \mathbb{R}^2$ $E = G = 1, F = 0, u = x, v = y$: From (15)

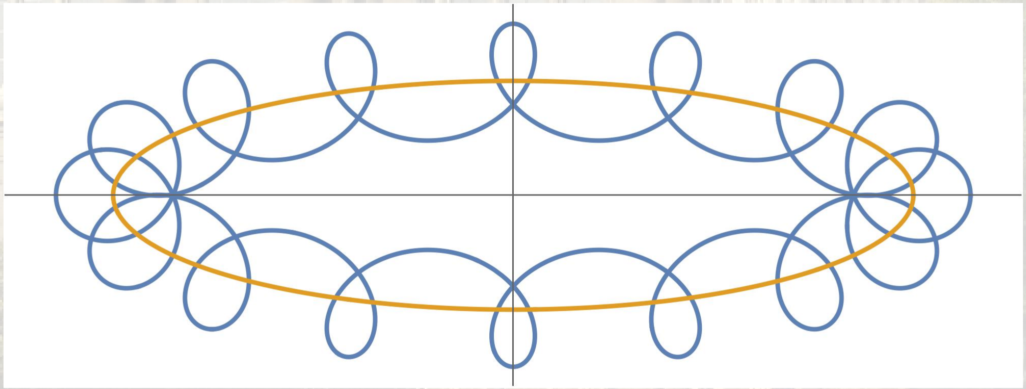
- $\Delta = 1$, $A = \varphi_u^2 + \varphi_v^2$, $B_1 = B_2 = 0$,

$$W = \frac{\varphi_{xx}\varphi_{x^2} - 2\varphi_{xy}\varphi_x\varphi_y + \varphi_{y^2}\varphi_x^2}{\varphi_x^2 + \varphi_y^2}$$

- and we get the "classical" Szebehely equation:[5]

$$\frac{\partial\varphi}{\partial x}\frac{\partial U}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial U}{\partial y} = 2\frac{\varphi_{xx}\varphi_{x^2} - 2\varphi_{xy}\varphi_x\varphi_y + \varphi_{y^2}\varphi_x^2}{\varphi_x^2 + \varphi_y^2}(E_0 - U(x,y)) \quad (15)$$

- prototype orbit to be analyzed



[5] Szebehely V G. *On the determination of the potential by satellite observations.* In: Proverbio E (Ed).) Proceedings Internat. Meeting on "Earth's Rotation by Satellite Observation", Bologna 1973, pp. 31-35.

Solution without energy prescription $E_0 = E_0(\varphi(u,v))$

▶ Use the auxiliary function

$$\alpha = \frac{1}{2W}\left(G\frac{\partial\varphi}{\partial u} - F\frac{\partial\varphi}{\partial v}\right); \quad \beta = \left(E\frac{\partial\varphi}{\partial v} - F\frac{\partial\varphi}{\partial u}\right); \quad \gamma = \frac{\partial\varphi}{\partial v}\left(\frac{\partial\varphi}{\partial u}\right)^{-1} \tag{16}$$

▶ Energy

$$E_0 = V + \alpha\frac{\partial U}{\partial u} + \beta\frac{\partial U}{\partial v}. \tag{17}$$

▶ Relations:

$$\frac{\partial E_0}{\partial u} = \frac{\partial E_0}{\partial \varphi}\frac{\partial\varphi}{\partial u}; \quad \frac{\partial E_0}{\partial v} = \frac{\partial E_0}{\partial \varphi}\frac{\partial\varphi}{\partial v} = \gamma\frac{\partial E_0}{\partial u}.^{6} \tag{18}$$

▶ The last identity and (17) lead to the hyperbolic equation

$$\alpha\gamma\,\frac{\partial^2 U}{\partial u^2} + (\beta\gamma - \alpha)\frac{\partial^2 U}{\partial u\partial v} - \beta\frac{\partial^2 U}{\partial v^2} + (\gamma + \gamma\alpha_u - \alpha_v)\frac{\partial U}{\partial u} + (\gamma\beta_u - \beta_v - 1)\frac{\partial U}{\partial v} = 0. \tag{19}$$

[6] Here we assume that the condition $\frac{\partial\varphi}{\partial u}\frac{\partial\varphi}{\partial v} \neq 0$ holds. For the degenerate case see [Bozis G, Mertens R 1985 *ZAMM* **65** 383].

The Drâmbã equation

- Describes motion in synodic references ($\omega = 1$):[7]

$$\ddot{u} - 2\dot{v} = \Omega_u, \quad \ddot{v} + 2\dot{u} = \Omega_v, \tag{20}$$

- Potential Ω includes the centrifugal term

$$\Omega = \frac{1}{2}\left(\dot{u}^2 + \dot{v}^2\right) - U(u,v) \Rightarrow \dot{u}^2 + \dot{v}^2 = 2\Omega - C \tag{21}$$

- In such a reference, (15) is given by[8]

$$\varphi_u \Omega_u + \varphi_v \Omega_v + W(2\Omega - C) \mp 2\sqrt{2\Omega - C}\sqrt{\varphi_u^2 + \varphi_v^2} = 0. \tag{22}$$

- "Disadvantage": truly non-linear equation.

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[7] Follow at once from

$$m\,\ddot{\mathbf{r}}' = \mathbf{F} - m\,\ddot{\mathbf{r}}_0 - m\,\dot{\boldsymbol{\omega}} \times \mathbf{r}' - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \dot{\mathbf{r}}'.$$

with the values $\mathbf{r}_0 = 0$, $\boldsymbol{\omega} = \mathbf{e}_3$ and $m = 1$.

[8] Originally proposed by Drâmbã C 1963 *St. Cerc. Astron. Acad. Rep. Pop. Rom.* **8** 7. See also Stavinschi M, Mioc V 1993 *Astron. Nachr.* **314** 91.

Genreric orbit problem on pseudo-Riemannian manifolds

- M manifold of dimension n, $g = g_{ij} dq^i \otimes dq^j$ nondegenerate metric tensor.
- $\mathfrak{X}(M)$ VF on M, $\Omega^1(M)$ 1-forms, ∇ Levi-Civita connection.
- $X \in \mathfrak{X}(M)$ is an integral of $\omega \in \Omega^1(M)$ if $\omega(X) = 0$.
- Given independent forms $\{w_1, \cdots, \omega_r\}$,[9] we consider

$$\mathcal{D}(X) = \{X \in \mathfrak{X}(M) \mid \omega_j(X) = 0,\ j = 1..r\} \tag{23}$$

- Extend to a (local) basis $\{w_1, \cdots, \omega_r, \theta_1, \cdots, \theta_{n-r}\} \subset \Omega^1(M)$, and take

$$\Xi = \omega_1 \wedge \omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_{n-r}, \tag{24}$$

- Volume form $\widehat{\Xi} = \Xi\left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \cdots, \frac{\partial}{\partial q_n}\right) \neq 0$
- NOTATION:

$$\{w_1, \cdots, \omega_r, \theta_1, \cdots, \theta_{n-r}\} = \{\Lambda_1, \cdots, \Lambda_n\} \tag{25}$$

[9] Distribution of order $n - r$.

- generic form of annihilating VF X: $(\frac{\partial}{\partial q^k} = \partial_{q^k})$

$$X = \det \begin{pmatrix} \omega_1(\partial_{q^1}) & \omega_1(\partial_{q^2}) & \cdots & \omega_1(\partial_{q^n}) & 0 \\ \omega_2(\partial_{q^1}) & \omega_2(\partial_{q^2}) & \cdots & \omega_2(\partial_{q^n}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_r(\partial_{q^1}) & \omega_r(\partial_{q^2}) & \cdots & \omega_r(\partial_{q^n}) & 0 \\ \theta_1(\partial_{q^1}) & \theta_1(\partial_{q^2}) & \cdots & \theta_1(\partial_{q^n}) & X^{r+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \theta_{n-r}(\partial_{q^1}) & \theta_{n-r}(\partial_{q^2}) & \cdots & \theta_{n-r}(\partial_{q^n}) & X^n \\ \partial_{q^1} & \partial_{q^2} & \cdots & \partial_{q^n} & 0 \end{pmatrix} \tag{26}$$

- Corresponding 1-form ω_X associated to X:

$$\omega_X = g_{jk} X^k dq^k \tag{27}$$

- Coboundary operator

$$d\omega_X = \frac{1}{2}\left(\frac{\partial p_j}{\partial q^k} - \frac{\partial p_k}{\partial q^j}\right) dq^j \wedge dq^k = \frac{1}{2} a^{jk} \Lambda_j \wedge \Lambda_k \tag{28}$$

► a^{jk} given in terms of $\widehat{\Xi}$

$$a^{jk} = (-1)^{j+k-1}\widehat{\Xi}^{-1} d\omega_X \wedge \Lambda_1 \wedge \cdots \wedge \widehat{\Lambda}_k \wedge \cdots \wedge \cdots \wedge \widehat{\Lambda}_j \wedge \cdots \wedge \Lambda_n \left(\frac{\partial}{\partial q^1}, \cdots, \frac{\partial}{\partial q^n} \right)$$

► Auxiliary condition:

$$X \rfloor d\omega_X \left(\frac{\partial}{\partial q^k} \right) = \Psi^j \Lambda_j \left(\frac{\partial}{\partial q^k} \right) \tag{29}$$

► System on M subjected to the constraint $\omega = F_k(\mathbf{q}, \dot{\mathbf{q}}) dq^k$ with $T = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j$:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}^k} \right) - \left(\frac{\partial T}{\partial q^k} \right) = Q_k = \omega \left(\frac{\partial}{\partial q^k} \right), \tag{30}$$

► If ω adopts the form

$$\omega = \frac{1}{2} \mathrm{d} \left(g(X, X) \right) + \sum_{j=1}^{r} \Psi^j \Lambda_j \tag{31}$$

► Then the trajectories of the system are those determined by the first order system with constraints $\Psi^j = 0$ for $j = r+1, \cdots, n$

$$\dot{\mathbf{q}} = X(\mathbf{q}) \tag{32}$$

▶ Formally, (30) with ω of the type (31) corresponds to a non-holonomic system constrained by the Pfaff form

$$\sum_{l=1}^{n} \Lambda_j \left(\frac{\partial}{\partial q^l} \right) \dot{q}^l = 0, \quad j = 1, \cdots r. \tag{33}$$

▶ If $\Lambda_j = d\varphi_j$ (Fröbenius), the system is holonomic.

▶ If $r = n - 1$, the most general force field giving rise to the orbits

$$\varphi_j = \alpha_j, \quad j = 1, \cdots r = n - 1. \tag{34}$$

is given by

$$\omega = \frac{1}{2}\mathrm{d}\left(g(X, X)\right) + \rho(\mathbf{q}) \sum_{j=1}^{n-1} a_{nj}\, \mathrm{d}\varphi_j \tag{35}$$

▶ The case (35) amplifies the methods of Szebehely, Mertens and Drâmbã.

▶ Applications: Nonholonomic systems.[10]

[10] Klein J 1962 *Ann. Inst. Fourier* **12** 1; Selig J M 1996 *Geometric Fundamentals of Robotics* (Springer, New York); Fedorov Yu N, Jovanovič B D 2004 *J. Nonlinear Sci.* **14** 341; Bloch A M 2005 *Nonholonomic Dynamics and Control Theory* (Springer, Berlin); Razavy M 2006 *Classical and Quantum Dissipative Systems* (Imperial College Press, London)

The Erugin approach

▶ Further Inverse problems in terms of quantitative or qualitative restrictions in the configuration or phase space (and its generalization in the Cartan sense)

▶ Two possibilities basing on "integral" manifolds: $\mathbf{q} = (q^1, \cdots, q^n)$

$$\omega_\mu(t, \mathbf{q}, \dot{\mathbf{q}}) = \lambda_\mu; \; 1 \leq \mu \leq n \tag{36}$$

• $\omega_\mu(t, \mathbf{q}, \dot{\mathbf{q}})$ compatible and independent.

1. Construct the equations of motion $\ddot{q}_\nu = \Phi(t, \mathbf{q}, \dot{\mathbf{q}}), \; 1 \leq \nu \leq n$ in accordance with (36). Analyze the existence conditions for (generalized) potentials.

2. Given a system $\ddot{q}_\nu = \Phi(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}), \; 1 \leq \nu \leq n$, determine the system parameters $\mathbf{v}$ compatible with (36) [system restauration]

▶ Both (and additional) possibilities reducible to a formulation in terms of DE (original solution due to Erugin)[11]

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[11] Erugin N P 1952 Prikl. Mat. Mekh. **16** 59; Galiullin A S 1960 *Some Questions about the Stability of Programmed Motion*, (KGU, Kazan); Dragovič V, Gajič B, Jovanovič B D 1998 *J. Phys. A: Math. Gen.* **31** 9861; Sadovskaia N, Ramírez R 2004 *J. Phys. A: Math. Gen.* **35** 3847; Ramírez R, Sadovskaia N 2007 *Rep. Math. Phys.* **60** 427; Borisov A V, Mamaev I S 2008 *Regul. Chaotic Dyn.* **13** 443; Soltakhanov Sh Kh, Yushkov M, Zegzhda S 2009 *Mechanics of Non-holonomic Systems. A New Class of Control Systems* (Springer, New York)

THE FIRST NOETHER THEOREM[12]

▶ Given the action integral

$$W = \int_{t_0}^{t_1} L\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\, dt, \qquad (37)$$

the total variation ΔW of W with respect to an infinitesimal transformation

$$\bar{t} = t + \Delta t; \; \overline{\mathbf{q}} = \mathbf{q} + \Delta \mathbf{q}$$

is given by

$$\Delta W = \int_{t_0}^{t_1} \sum_{k=1}^{N} \frac{\delta L}{\delta q_k} \left(\Delta q_k - \dot{q}_k \Delta t\right) dt + \int_{t_0}^{t_1} \frac{d}{dt} \left[L \Delta t + \sum_{k=1}^{N} \left(\frac{\partial L}{\partial \dot{q}_k} \Delta q_k - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \Delta t \right) \right] dt, \qquad (38)$$

with

$$\frac{\delta L}{\delta q_k} = \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right)$$

the variational derivatives of L with respect to q_k

[12]Noether E 1918 *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **1918** 235; Bessel-Hagen E 1921 *Math. Annalen* **84** 258; Funk P 1962 *Variationsrechnung und ihre Anwendung in Physik und Technik* (Berlin: Springer)

- r-parameter group $\{\epsilon_1, \cdots, \epsilon_r\}$
- Invariance of W up to a divergence:

$$\bar{t} = t + \sum_{\sigma=1}^{r} \epsilon_\sigma \xi^\sigma (t, \mathbf{q}, \dot{\mathbf{q}})\,;\; \bar{q}_k = q_k + \sum_{\sigma=1}^{r} \epsilon_\sigma \eta_k^\sigma (t, \mathbf{q}, \dot{\mathbf{q}})\,,\; k = 1, \cdots, N \tag{39}$$

- There exist functions (gauge terms) $V^\sigma (t, \mathbf{q}, \dot{\mathbf{q}})$ for $1 \leq \sigma \leq r$ such that

$$\Delta W \left[\sum_{\sigma=1}^{r} \epsilon_\sigma \xi^\sigma, \sum_{\sigma=1}^{r} \epsilon_\sigma \eta^\sigma \right] = \int_{t_0}^{t_1} \frac{d}{dt} \left(\sum_{\sigma=1}^{r} \epsilon_\sigma V^\sigma \right) \tag{40}$$

- Absolute invariance: $V^\sigma = 0$ for all σ.

► The Noether theorem does **not** impose the symmetry generator to be a mere transformation of the (extended) configuration space, as has been "popularized" after Hill's paper.[13]

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[13] Hill E L 1951 *Rev. Mod. Phys.* **23** 253; Komorowski J 1968 *Studia Math.* **29** 261; Khukhunashvili Z V 1968 *Soviet Phys.* **11** 1; Rund H 1972 *Utilitas Math.* **2** 205; Havas P 1973 *Acta Phys. Austriaca* **38** 145; Logan J D 1973 *Aequat. Math.* **9** 210; Logan J D 1974 *J. Math. Anal. Appl.* **42** 191; Logan J D, Blakeslee J S 1975 *J. Math. Phys.* **16** 1374; Kosmann-Schwarzbach Y 2011 *The Noether Theorems* (New York: Springer Verlag)

► **First Noether Theorem.** *If W is invariant up to a divergence under the r-parameter group of transformations (39), then r linearly independent combinations of the variational derivatives become divergences, i.e.,*

$$-\sum_{k=1}^{N} \frac{\delta L}{\delta q_k} (\eta_k^\sigma - \dot{q}_k \xi^\sigma) = \frac{d}{dt} \left[\left(L - \sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right) \xi^\sigma + \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \eta_k^\sigma - V^\sigma \right], \quad (41)$$

where $\sigma = 1, \cdots, r$.

► Converse also holds. If r linearly independent combinations of the variational derivatives are divergences, W is invariant with respect to an r-parameter group of transformations.

► Variation problem: $\frac{\delta L}{\delta q_k} = 0$ $(1 \leq k \leq N)$ implies that

$$\Psi^\sigma \doteq \left[L - \sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right] \xi^\sigma (t, \mathbf{q}, \dot{\mathbf{q}}) + \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \eta_k^\sigma (t, \mathbf{q}, \dot{\mathbf{q}}) - V^\sigma (t, \mathbf{q}, \dot{\mathbf{q}}), \quad \sigma = 1, \cdots, r \quad (42)$$

are constants of the motion.

- Expanding $\frac{d}{dt}\Psi^\sigma$ leads to

$$\frac{d\Psi^\sigma}{dt} = \frac{dL}{dt}\xi^\sigma + L\frac{d\xi^\sigma}{dt} - \sum_{k=1}^{N}\left(\frac{d}{dt}\left(\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}\right)\xi^\sigma - \dot{q}_k\frac{\partial L}{\partial \dot{q}_k}\frac{d\xi^\sigma}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\eta_k^\sigma + \frac{\partial L}{\partial \dot{q}_k}\frac{d\eta_k^\sigma}{dt}\right) - \frac{dV^\sigma}{dt}$$
$$= \xi^\sigma\frac{d}{dt}\left(L - \sum_{k=1}^{N}\dot{q}_k\frac{\partial L}{\partial \dot{q}_k}\right) + L\frac{d\xi^\sigma}{dt} + \sum_{k=1}^{N}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{\eta}_k^\sigma + \eta_k^\sigma\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right) - \frac{dV^\sigma}{dt}$$
$$= \xi^\sigma\frac{\partial L}{\partial t} + \sum_{k=1}^{N}\left(\eta_k^\sigma\frac{\partial L}{\partial q_k} + \frac{\partial L}{\partial \dot{q}_k}\dot{\eta}_k^\sigma\right) + L\frac{d\xi^\sigma}{dt} - \frac{dV^\sigma}{dt} = 0.$$

- first prolongation

$$\dot{\mathbf{X}}_\sigma = \xi^\sigma\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\frac{\partial}{\partial t} + \sum_{k=1}^{N}\left\{\eta_k^\sigma\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\frac{\partial}{\partial q_k} + \dot{\eta}_k^\sigma\left(t, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right)\frac{\partial}{\partial \dot{q}_k}\right\} \tag{43}$$

- For infinitesimal symmetry generators $\mathbf{X}_\sigma\ (\epsilon_\sigma)$, invariance is given by

$$\dot{\mathbf{X}}_\sigma\left(L\right) - \mathbf{A}\left(\xi^\sigma\right)L - \mathbf{A}\left(V^\sigma\right) = 0. \tag{44}$$

PARTICULAR TYPES:

- Point symmetry if

$$\frac{\partial \xi^\sigma}{\partial \dot{q}_k} = \frac{\partial \eta_k^\sigma}{\partial \dot{q}_k} = 0,\ 1 \le k \le N,\ 1 \le \sigma \le r \tag{45}$$

- "Dynamical" symmetry if (45) does not hold.
- Variational symmetry if $\xi(t, \mathbf{q}) = V(t, \mathbf{q}) = 0$.

► For kinetic Lagrangians $L = \sum_{j,k=1}^{N} g_{jk}(\mathbf{q})\, \dot{q}_j \dot{q}_k$, symmetry generators $\mathbf{X}$ are Killing vectors of:

$$\bar{g} = \sum_{j,k=1}^{N} g_{jk}(\mathbf{q})\, dq_j dq_k$$

► Related to variational symmetries.

▶ The necessary and sufficient condition for a Killing vector of

$$\overline{g} = \sum_{j,k=1}^{N} g_{jk}\left(\mathbf{q}\right) dq_j dq_k$$

to exist: there is some index s_0 such that

$$\frac{\partial g_{jk}}{\partial q_{s_0}} = 0$$

for all $1 \leq j, k \leq N$.

▶ Lie algebra of Killing vectors is a subalgebra of $\mathcal{L}_{NS}$ (L of kinetic type).

▶ N-dimensional linearizable Lagrangian systems: $\mathcal{L}_{NS}$ is isomorphic to the unextended Schrödinger algebra $S(N)$.[14]

[14] González López A 1988 *J. Math. Physics* **29** 1097; Gorringe V. M, Leach P. G. L. 1988 *Quaest. Math.* **11** 95; Olver P J 1995 *Equivalence, Invariance and Symmetry* (Cambridge: Cambridge University Press; Wafo Soh C 2010. *Commun. Nonlinear Sci. Numer. Simulat.* **15** 139; RCS 2011 *Comm. Nonlin. Sci. Num. Sim.* **16** 3015; RCS 2012 *Comm. Nonlin. Sci. Num. Sim.* **17** 1178; Moyo S., Meleshko S. V., Oguis G. F 2013 *Comm. Nonlinear Science Num. Simulation* **18** 2972; RCS 2014 *Comm. Nonlinear Science Num. Simulation* **19** 2602; Gray R. J. 2014 *Proc. R. Soc. A* **470** 20130779;)

INVERSE PROBLEM INVOLVING NOETHER-SYMMETRY PRESERVATION:[15]

▶ Given $L(t, \mathbf{q}, \dot{\mathbf{q}})$ with Lie algebra $\mathcal{L}_{NS}$, and given $\mathfrak{g} \subset \mathcal{L}_{NS}$, determine perturbations

$$\widehat{L}(t, \mathbf{q}, \dot{\mathbf{q}}) = L(t, \mathbf{q}, \dot{\mathbf{q}}) + \varepsilon R(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{46}$$

▶ subjected to the conditions

- det $\left(\frac{\partial^2 \widehat{L}}{\partial q^i \partial q^j}\right) = \text{cons} \neq 0$ (regularity)
- $\mathfrak{g} \simeq \widehat{\mathcal{L}}_{NS}$ with identical generators (symmetry breaking)
- Divergences $V(t, \mathbf{q})$ are preserved
- $\widehat{\psi}_X = \psi_X + \varepsilon \Psi(R(t, \mathbf{q}, \dot{\mathbf{q}}))$ for the constants of the motion
- The system $\widehat{L}$ is integrable in the Liouville sense

▶ Existence of interesting perturbations depends heavily on $\mathfrak{g}$

▶ Possible solutions without Liouville integrability.

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[15] RCS, Guerón J 2013 *Acta Appl. Math.* **127** 105; RCS 2013 *Int. J. Geom. Meth. Phys.* **10** 130006; RCS 2014 *J. Math. Phys.* **55** 042904; RCS 2014 *Comm. Nonlin. Sci. Num. Sim.* **19** 2602; RCS 2016 *Acta Mech.* **227** 1941; RCS 2018 *Acta Mech.* **229** 211

Generalized Ermakov-Ray-Reid systems as perturbations of the oscillator

▶ Damped oscillator in $N = 2$: $[g_1(t), g_2(t)$ arbitrary$]$

$$\ddot{q}_i + g_1(t)\,\dot{q}_i + g_2(t)\,q_i = 0,\ \ i = 1,2 \tag{47}$$

• admissible Lagrangian:

$$L = \frac{1}{2}\varphi(t)\left(\dot{q}_1^2 + \dot{q}_2^2 - g_2(t)\left(q_1^2 + q_2^2\right)\right). \tag{48}$$

• Noether symmetries: $X = \xi(t,\mathbf{q})\frac{\partial}{\partial t} + \eta_j(t,\mathbf{q})\frac{\partial}{\partial q_j}$ $(1 \leq j \leq 2, k \neq j)$:

$$\xi(t,\mathbf{q}) = \xi(t);\ \ \eta_j(t,\mathbf{q}) = \frac{1}{2}\left(\dot{\xi}(t) - g_1(t)\,\xi(t)\right)q_j + \lambda_j^k q_k + \psi_j(t), \tag{49}$$

• $\xi(t)$ satisfies the ODE

$$\frac{d^3\xi}{dt^3} + \left(4g_2(t) - g_1^2(t) - 2\frac{dg_1}{dt}\right)\frac{d\xi}{dt} + \left(2\frac{dg_2}{dt} - \left(\frac{dg_1}{dt}\right)^2 - g_1(t)\,\frac{dg_1}{dt}\right)\xi = 0. \tag{50}$$

- The more general $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$-preserving perturbation is

$$\widetilde{L}_m = \frac{1}{2}\varphi\left(t\right)\left(\dot{q}_1^2 + \dot{q}_2^2 - g_2\left(t\right)\left(q_1^2 + q_2^2\right)\right) + G_{\varepsilon,m}\left(t, \mathbf{q}, \dot{\mathbf{q}}\right), \tag{51}$$

- where

$$G_{\varepsilon}\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) = \varepsilon\, F\left(\frac{q_2}{q_1}, \varphi\left(t\right)\left(\dot{q}_2 q_1 - \dot{q}_1 q_2\right)\right) q_1^{-2}\varphi^{-2}\left(t\right). \tag{52}$$

- If $g_1\left(t\right) = 0$, G reduces to

$$G_{\varepsilon}\left(\mathbf{q}, \dot{\mathbf{q}}\right) = \varepsilon\, F\left(\frac{q_2}{q_1}, \left(\dot{q}_2 q_1 - \dot{q}_1 q_2\right)\right) q_1^{-2}. \tag{53}$$

- In any case, $(\widetilde{\mathcal{L}}_m)_{NS} \simeq \mathfrak{sl}(2, \mathbb{R})$.
- Auxiliary variables: $r = q_2 q_1^{-1}$, $W = q_1\dot{q}_2 - \dot{q}_1 q_2$

$$\widetilde{L} = \frac{\varphi\left(t\right)}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 - g_2\left(t\right)\left(q_1^2 + q_2^2\right)\right) - \frac{\varepsilon}{q_1^2\varphi\left(t\right)} F\left(r, \varphi\left(t\right) W\right) \tag{54}$$

► Equations of the motion:

$$\ddot{q}_1 + g_1(t)\,\dot{q}_1 + g_2(t)\,q_1 + \frac{2\varepsilon\dot{r}}{\varphi(t)\,q_1}\frac{\partial F}{\partial W} + \frac{\varepsilon\, r}{\varphi(t)^2\,q_1^3}\left(\varphi(t)\,W\frac{\partial^2 F}{\partial r\partial W} - \frac{\partial F}{\partial r}\right)$$

$$-\frac{2\varepsilon\, F}{\varphi(t)^2\,q_1^3} + \frac{\varepsilon\, r\left(\dot{W} + g_1(t)\,W\right)}{q_1}\frac{\partial^2 F}{\partial W^2} = 0,$$

$$\ddot{q}_2 + g_1(t)\,\dot{q}_2 + g_2(t)\,q_2 + \varepsilon\frac{\frac{\partial F}{\partial r} - \varphi(t)\,W\frac{\partial^2 F}{\partial r\partial W}}{\varphi(t)^2\,q_1^3} - \varepsilon\frac{\left(\dot{W} + g_1(t)\,W\right)}{q_1}\frac{\partial^2 F}{\partial W^2} = 0.$$

► “Classical” Ermakov-Ray-Reid systems:[16]

$$\ddot{q}_1 + \omega^2(t)\,q_1 - \frac{1}{q_2^3}\frac{\partial F_1}{\partial r} + \frac{W}{q_2^3}\frac{\partial^2 F_1}{\partial r\partial W} + \frac{\dot{W}}{q_2}\frac{\partial^2 F_1}{\partial W^2} = 0, \qquad (55)$$

$$\ddot{q}_2 + \omega^2(t)\,q_2 + \frac{1}{q_2^2 q_1}\frac{\partial G_1}{\partial r} - \frac{W}{q_2^2 q_1}\frac{\partial^2 G_1}{\partial r\partial W} - \frac{\dot{W}}{q_1}\frac{\partial^2 G_1}{\partial W^2} = 0, \qquad (56)$$

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[16] Ray J R, Reid J L 1979 *Phys. Lett. A* **71** 317; Ray J R, Reid J L 1981 *J. Math. Phys.* **22** 91; Reid JL, Ray JR 1982 *J. Phys. A: Math. Gen.* **15** 2751; Goedert J 1989 *Phys. Lett. A* **136** 391; Rogers C, Hoenselaers C, Ramgulam U 1993 *Ermakov structure in 2+1-dimensional systems. Canonical reduction* In: Ibragimov, N. H., Torrisi, M., Valenti, A. (eds.) Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, pp. 317 (Kluwer, Amsterdam); Govinder K S, Leach P G L 1994 *Phys. Lett. A* **186** 391; Haas F, Goedert, J 2001 *Phys. Lett. A* **279** 181.

► Helmholtz conditions:[17]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right) - \frac{\partial L}{\partial q^j} = f_{ij}\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\left(\ddot{q}^i - \omega^i\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\right),\ 1 \leq i, j \leq N \tag{57}$$

• The system (55)-(56) is Lagrangian

$$\frac{\partial^2 G_1}{\partial W^2} - r^2 \frac{\partial^2 F_1}{\partial W^2} = 0, \tag{58}$$

$$3\frac{\partial F_1}{\partial r} + \frac{1}{r^2}\frac{\partial G_1}{\partial r} - 3w\frac{\partial^2 F_1}{\partial r \partial W} - \frac{W}{r^2}\frac{\partial^2 G_1}{\partial r \partial W} + r\frac{\partial^2 F_1}{\partial r^2} - \frac{1}{r}\frac{\partial^2 G_1}{\partial r^2} - rw\frac{\partial^3 F_1}{\partial r^2 \partial W} + \frac{W}{r}\frac{\partial^3 G_1}{\partial r^2 \partial W} = 0 \tag{59}$$

► CONSEQUENCE: For $g_1(t) = 0$ and $g_2(t) = \omega^2(t)$, any perturbation of the damped oscillator preserving the $\mathfrak{sl}(2, \mathbb{R})$-subalgebra corresponds to a generalized ERR system (55)-(56). Conversely, any Hamiltonian ERR system is a perturbation of the oscillator.

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[17] Inverse problem of Lagrangian mechanics. See *e.g.* von Helmholtz H 1887 *J. reine angew. Math.* **100** 137; Mayer A 1896 *Ber. kön. Ges. Wiss. Leipzig* **519** 1; Davis D R 1928 *Trans. Amer. Math. Soc.* **30** 710; Havas P 1956 *Bull. Am. Phys.* **1** 12; Havas P 1957 *Suppl. Nuovo Cimento* **5** 363; Vujanovic B 1971 *ATAA J.* **9** 131; Havas P 1973 *Acta Phys. Austriaca* **38**, 145; Djukič Dj S, Vujanovič B 1975 *Acta Mech.* **23** 17; Engels E 1975 *Nuovo Cimento* **26**B 481; Engels E 1978 *Hadronic J.* **1** 465; Santilli R M 1978 *Foundations of Theoretical Mechanics vol. I* (Springer, New York);

Free Lagrangian in $\mathbb{R}^N$

- Free system determined by $L = \dot{q}^i \dot{q}_i$.
- Noether symmetries $\mathfrak{sl}\left(2,\mathbb{R}\right) \oplus \mathfrak{so}\left(N\right)$:

$$X_1 = t^2 \frac{\partial}{\partial t} + \sum_{i=1}^{N} t\, q_i \frac{\partial}{\partial q_i} + \sum_{i=1}^{N} \left(q_i - t\, \dot{q}_i\right) \frac{\partial}{\partial \dot{q}_i}, \quad X_3 = \frac{\partial}{\partial t},$$

$$X_2 = -\left[X_1, X_3\right], \quad X_{i,j} = q_j \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial q_j},\ 1 \leq i, < j \leq N.$$

- $\mathfrak{so}(N)$-part corresponds to Killing vectors.
- most general perturbation of$L = \dot{q}^i \dot{q}_i$ preserving $\mathfrak{so}(N) \oplus \mathfrak{sl}(2,\mathbb{R})$:

$$\widetilde{L} = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \cdots + \dot{q}_N^2\right) - \varepsilon\, \frac{F\left(\left(\sum_{i=1}^{N} q_i^2\right)\left(\sum_{i=1}^{N} \dot{q}_i^2\right) - \left(\sum_{i=1}^{N} q_i \dot{q}_i\right)^2\right)}{\left(q_1^2 + q_2^2 + \cdots + q_N^2\right)}, \tag{60}$$

- Invariance with respect to other symmetries $\Longrightarrow$ cyclic coordinates.
- Non-linearizable systems: $\mathcal{L}'_{NS} \simeq \mathfrak{so}(m) \oplus \mathfrak{sl}(2,\mathbb{R}) < \mathfrak{so}(N) \oplus \mathfrak{sl}(2,\mathbb{R})$

► Auxiliary functions:

$$J_1 = \sum_{i=1}^{m} q_i \dot{q}_i, \;\; J_2 = \sum_{i=1}^{m} q_i^2, \;\; J_3 = \sum_{i=1}^{m} \dot{q}_i^2 \tag{61}$$

► Auxiliary variables:

$$z_j = \frac{q_{m+j}}{\sqrt{J_2}}, \;\; z_{N-m+j} = \frac{\dot{q}_{m+j}\, J_2 - q_{m+j}\, J_1}{\sqrt{J_2}}, \; 1 \le j \le N-m. \tag{62}$$

► Perturbations:

$$\widetilde{L}_m = \frac{1}{2}\left(\dot{q}_1^2 + \cdots + \dot{q}_N^2\right) + G_{\varepsilon,m}\left(t, \mathbf{q}, \dot{\mathbf{q}}\right), \tag{63}$$

► where

$$G_{\varepsilon,m}\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) = \varepsilon \frac{F\left(J_2 J_3 - J_1^2, z_1, \cdots, z_{2N-2m}\right)}{J_2}. \tag{64}$$

► $\mathcal{L}'_{NS}$-preserving potentials:

$$G_{\varepsilon,m}\left(t, \mathbf{q}, \right) = \varepsilon\, \frac{F\left(z_1, \cdots, z_{N-m}\right)}{J_2}. \tag{65}$$

EXAMPLE: $\mathcal{L}'_{NS} \simeq \mathfrak{so}(2) \oplus \mathfrak{sl}(2,\mathbb{R}) < \mathfrak{so}(3) \oplus \mathfrak{sl}(2,\mathbb{R})$ IN $N = 3$

- Generic perturbation in cylindric reference:

$$\widetilde{L} = \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{\xi}^2\right) - \frac{\varepsilon}{\rho^2} F\left(\frac{\xi}{\rho}\right), \tag{66}$$

- θ cyclic $\Longrightarrow J = \rho^2\dot{\theta} = \alpha =$ cons.
- Reduction:

$$\widetilde{L}' = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right) + \frac{\alpha^2}{2\rho^2} - \frac{\varepsilon}{\rho^2} F\left(\frac{\xi}{\rho}\right). \tag{67}$$

- (67) is a perturbation of $L_0 = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right)$ [preserves $\mathfrak{sl}(2,\mathbb{R})$]
- Constants of the motion:

$$H_r = \frac{1}{2}\left(\dot{\rho}^2 + \dot{\xi}^2\right) + \frac{1}{\rho^2}\left(\varepsilon F\left(\frac{\xi}{\rho}\right) - \frac{\alpha^2}{2}\right), \tag{68}$$

$$I_1 = \frac{1}{2}\left(\xi\dot{\rho} - \dot{\xi}\rho\right)^2 + \int^{\xi\rho^{-1}} \left(z\left(2\varepsilon F(z) - \alpha^2\right) + \varepsilon\left(1 + z^2\right)\frac{dF}{dz}\right) dz. \tag{69}$$

- Question: Can perturbations alter the curvature?

RIEMANN CURVATURE TENSOR

- M Riemannian manifold. Regular metric tensor $\overline{g} = \sum_{i,j=1}^{N} g_{ij}\left(\mathbf{q}\right) dq_i dq_j$.

$$\Gamma_{ij}^{h} = \frac{1}{2}\sum_{k=1}^{N} g^{hk}\left\{\frac{\partial g_{jk}}{\partial q_i} + \frac{\partial g_{ik}}{\partial q_j} - \frac{\partial g_{ij}}{\partial q_k}\right\}. \tag{70}$$

- Geodesic (local) equations

$$\ddot{q}_i + \sum_{j,k=1}^{N} \Gamma_{jk}^{i}\dot{q}_j\dot{q}_k = 0. \tag{71}$$

- (71) correspond to the equations of motion of the kinetic Lagrangian

$$L = \sum_{i,j=1}^{N} g_{ij}\left(\mathbf{q}\right)\dot{q}_i\dot{q}_j.$$

► Riemann tensor (in (0,4) tensor form)

$$R_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial q_j \partial q_k} - \frac{\partial^2 g_{jl}}{\partial q_i \partial q_k} - \frac{\partial^2 g_{ik}}{\partial q_j \partial q_l} + \frac{\partial^2 g_{jk}}{\partial q_i \partial q_l}\right) + q_{pq}\left(\Gamma^p_{li}\Gamma^q_{kj} - \Gamma^p_{ki}\Gamma^q_{lj}\right) = g_{ih} R^h_{jkl}.$$

► Ricci $(0,2)$-tensor

$$R_{ij} = \sum_{p=1}^{N} R^p_{ipj} = \sum_{k=1}^{N}\left(\frac{\partial \Gamma^k_{ji}}{\partial q_k} - \frac{\partial \Gamma^k_{ki}}{\partial q_j}\right) + \sum_{k,p=1}^{N}\left(\Gamma^k_{kp}\Gamma^p_{ji} - \Gamma^k_{jp}\Gamma^p_{ki}\right).$$

CONSTANTS OF THE MOTION OF PERTURBATIONS

- Let $J\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ be an invariant of $L = \sum_{i,j=1}^{N} g_{ij}\left(\mathbf{q}\right)\dot{q}_i\dot{q}_i$.
- $J\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ is a constant of the motion of $\widetilde{L} = L(t, \mathbf{q}, \dot{\mathbf{q}}) - U\left(t, \mathbf{q}\right)$ whenever

$$\sum_{k=1}^{N} (-1)^k \left\{ \sum_{l=1}^{N} (-1)^l \left|\mathbf{g}_{(k,l)}\right| \frac{\partial U}{\partial q_l} \right\} \frac{\partial J}{\partial \dot{q}_k} = 0, \tag{72}$$

is satisfied, where $\mathbf{g}_{(k,l)}$ denotes the minor of $\mathbf{g}$ obtained by deletion of the k^{th} row and l^{th} column.

- Lagrange equations for $\widetilde{L}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0,\ 1 \leq i \leq N. \tag{73}$$

- For any index, the identity

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{N} \left(\frac{dg_{ij}}{dt}\dot{q}_j - \frac{1}{2}\frac{\partial g_{ij}}{\partial q_i}\dot{q}_j + g_{ij}\ddot{q}_j \right) = 0 \tag{74}$$

holds (normal form of equations of motion of L).

► Normal equations of motion of $\widetilde{L}$

$$\widetilde{\omega}_i = \omega_i + \frac{(-1)^i}{|\mathbf{g}|} \sum_{l=1}^{N} (-1)^l \left|\mathbf{g}_{(i,l)}\right| \frac{\partial U}{\partial q_l},\ 1 \leq i \leq N. \tag{75}$$

► Now $J(t, \mathbf{q}, \dot{\mathbf{q}})$ is a constant of the motion of L implies that

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \sum_{i=1}^{N} \left(\dot{q}_i \frac{\partial J}{\partial q_i} + \omega_i \frac{\partial J}{\partial \dot{q}_i} \right) = 0. \tag{76}$$

► Supposed that $J(t, \mathbf{q}, \dot{\mathbf{q}})$ is also an invariant of $\widetilde{L}$:

$$\frac{dJ}{dt} = \frac{\partial J}{\partial t} + \sum_{i=1}^{N} \dot{q}_i \frac{\partial J}{\partial q_i} + \left(\omega_i + \frac{(-1)^i}{|\mathbf{g}|} \sum_{l=1}^{N} (-1)^l \left|\mathbf{g}_{(i,l)}\right| \frac{\partial U}{\partial q_l} \right) \frac{\partial J}{\partial \dot{q}_i} = 0. \tag{77}$$

► By (76), the latter reduces to the PDE for $U(t, \mathbf{q})$:

$$\sum_{i=1}^{N} \frac{(-1)^i}{|\mathbf{g}|} \left(\sum_{l=1}^{N} (-1)^l \left|\mathbf{g}_{(i,l)}\right| \frac{\partial U}{\partial q_l} \right) \frac{\partial J}{\partial \dot{q}_i} = 0. \tag{78}$$

- This equation is separable after the variables $\dot{q}_i$.
- EXAMPLE:

$$L = \frac{1}{2}\left(\dot{q}_1^2 + q_1^2\left(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right)\right)$$

in spherical coordinates.

- Quadratic invariant $J = q_1^4\left(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right)$.
- Only nonzero entries of the matrix $\mathbf{g}$ are $g_{11} = 1$, $g_{22} = q_1^2$, $g_{33} = q_1^2 \sin^2 q_2$
- Only nonvanishing minors are

$$\left|\mathbf{g}_{(1,1)}\right| = q_1^4 \sin^2 q_2, \; \left|\mathbf{g}_{(2,2)}\right| = q_1^2 \sin^2 q_2, \; \left|\mathbf{g}_{(3,3)}\right| = q_1^2.$$

- Resulting PDE for $L - U(t, \mathbf{q})$:

$$2q_1^6 \sin^2 q_2 \left(\dot{q}_2 \frac{\partial U}{\partial q_2} + \dot{q}_3 \frac{\partial U}{\partial q_3}\right) = 0. \tag{79}$$

- Separable equation with solution potentials $U(\mathbf{q}) = V(t, q_1)$.

Systems containing dissipative terms

- Lagrangian $L = \sum_{i,j=1}^{N} g_{ij}\left(\mathbf{q}\right)\dot{q}_i\dot{q}_j$.
- $J\left(t,\mathbf{q},\dot{\mathbf{q}}\right)$ invariant of L homogeneous of order α in the velocities $\dot{\mathbf{q}}$.
- $\widetilde{J} = f\left(t\right)^{\alpha} J\left(t,\mathbf{q},\dot{\mathbf{q}}\right)$ is an invariant of $\widetilde{L} = f\left(t\right)L$.
- Normal form $\ddot{q}_i = \widetilde{\omega}_i\left(t,\mathbf{q},\dot{\mathbf{q}}\right)$ of the EM of $\widetilde{L}$

$$\widetilde{\omega}_i\left(t,\mathbf{q},\dot{\mathbf{q}}\right) = \omega_i\left(t,\mathbf{q},\dot{\mathbf{q}}\right) - \frac{d}{dt}\left(\ln f\left(t\right)\right)\dot{q}_i,\ 1 \leq i \leq N. \tag{80}$$

- Total differential of $\widetilde{J}$

$$\frac{d\widetilde{J}}{dt} = \cdots = f\left(t\right)^{\alpha}\frac{dJ}{dt} + \dot{f}\left(t\right)f\left(t\right)^{\alpha-1}\left(\alpha J - \sum_{i=1}^{N}\dot{q}_i\frac{\partial J}{\partial\dot{q}_i}\right) = 0. \tag{81}$$

- First term vanishes (invariance of J).
- Second term is zero by the Euler theorem on homogeneous functions.

► Extension to Lagrangians with a potential are possible by considering invariants as a sum of homogeneous functions in the $\dot{\mathbf{q}}$ of decreasing degree, and modifying it according to these orders.

► $\mathbf{X}$ Killing vector of

$$\overline{ds^2} = \sum_{i,j=1}^{N} g_{ij}(\mathbf{q})\, dq_i\, dq_j$$

associated to the kinetic Lagrangian $L = \sum_{i,j=1}^{N} g_{ij}(\mathbf{q})\, \dot{q}_i \dot{q}_j$.

► $\mathbf{X}$ is a Noether point symmetry of $\widetilde{L} = f(t)\, L$ for any non-zero function $f(t)$.

As $\mathbf{X}$ is a Killing vector, we have that $\xi(t, \mathbf{q}) = V(t, \mathbf{q}) = 0$ and $\frac{\partial \eta_i}{\partial t} = 0$ for $1 \leq i \leq n$ in the symmetry condition (44). Hence

$$\dot{\mathbf{X}}\left(\widetilde{L}\right) = \dot{\mathbf{X}}\left(f(t)\, L\right) = f(t) \sum_{i=1}^{N} \left(\eta_i \frac{\partial L}{\partial q_i} + \dot{\eta}_i \frac{\partial L}{\partial \dot{q}_i} \right) = f(t)\, \dot{\mathbf{X}}(L) = 0.$$

PLANE PERTURBATIONS

- Most obvious starting case

$$L_n = \frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2 \dot{q}_2^2\right) \tag{82}$$

- $n = 0$: L_0 free Lagrangian
- $n \neq 0$: can be seen as a position-dependent mass system.
- Symmetry generators $\mathbf{X} = \xi\left(t, \mathbf{q}\right)\frac{\partial}{\partial t} + \eta^1\left(t, \mathbf{q}\right)\frac{\partial}{\partial q_1} + \eta^2\left(t, \mathbf{q}\right)\frac{\partial}{\partial q_2}$

$$\begin{aligned}
\xi\left(t, \mathbf{q}\right) &= b_1 t^2 + b_2 t + b_3, \\
\eta^1\left(t, \mathbf{q}\right) &= \frac{\left(2b_1 t + b_2\right)}{n+2} q_1 + q_1^{-\frac{n}{2}}\left\{\left(C_{11} t + C_{21}\right)\sin\left(\frac{n+2}{2} q_2\right) + \left(C_{12} t + C_{22}\right)\cos\left(\frac{n+2}{2} q_2\right)\right\}, \\
\eta^2\left(t, \mathbf{q}\right) &= -q_1^{-\frac{n+2}{2}}\left\{\left(C_{12} t + C_{22}\right)\sin\left(\frac{n+2}{2} q_2\right) + \left(C_{11} t + C_{21}\right)\cos\left(\frac{n+2}{2} q_2\right)\right\} + b_4.
\end{aligned}$$

- Gauge term

$$V\left(t, \mathbf{q}\right) = \frac{2}{n+2} q_1^{\frac{n+2}{2}}\left(C_{11}\sin\left(\frac{n+2}{2} q_2\right) + C_{12}\cos\left(\frac{n+2}{2} q_2\right)\right) + \frac{2b_1}{n+2} q_1^{n+2},$$

- $\mathcal{L}_{NS}$ isomorphic $S(2)$
- Levi part of isomorphic to $\mathfrak{sl}(2, \mathbb{R})$

$$X_1 = t^2 \frac{\partial}{\partial t} + \frac{2t\, q_1}{n+2} \frac{\partial}{\partial q_1}, \;\; X_2 = t \frac{\partial}{\partial t} + \frac{q_1}{n+2} \frac{\partial}{\partial q_1}, \;\; X_3 = \frac{\partial}{\partial t}. \tag{83}$$

- Hamiltonian H constant of the motion.
- Symmetries corresponding to C_{21}, C_{22}, b_4 variational.
- Most general term quadratic in $\dot{\mathbf{q}}$

$$R(t, \mathbf{q}, \dot{\mathbf{q}}) = q_1^{(n+2)} S(q_2)\, \dot{q}_2^2 + T(q_2)\, \dot{q}_2 + U(q_2)\, q_1^{-(n+2)}, \tag{84}$$

- $T(q_2) = 0$, as a total time derivative.
- Regularity ensured if

$$1 + 2S(q_2) \neq 0.$$

- Riemann curvature obviously zero.

▶ For generic functions $S(q_2)$ and $U(q_2) \neq 0$

$$L_{n,\varepsilon} = \frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\left(1 + 2S\left(q_2\right)\right)\dot{q}_2^2\right) + U\left(q_2\right) q_1^{-(n+2)}$$

is a solution to the inverse problem with $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$.

▶ Constants of the motion: Hamiltonian and

$$J_2 = \frac{q_1^{2n+4}}{2}\left(1 + 2S\left(q_2\right)\right)\dot{q}_2^2 + U\left(q_2\right). \tag{85}$$

Introduction of time-dependent factor

$$\widetilde{L}_n = \varphi\left(t\right)\frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2\right), \; n \neq -2. \tag{86}$$

▶ System still linearizable with $\dim \mathcal{L}_{NS} = 8$.

► Symmetries

$$\begin{aligned}
\xi\left(t,\mathbf{q}\right) = &\quad G(t),\\
\eta^{1}\left(t,\mathbf{q}\right) = &\quad q_1^{-\frac{n}{2}}\left\{\sin\left(\tfrac{n+2}{2}q_2\right)\left(C_1 + C_2\int\tfrac{dt}{\varphi(t)}\right) + \cos\left(\tfrac{n+2}{2}q_2\right)\left(C_3 + C_4\int\tfrac{dt}{\varphi(t)}\right)\right\} + \tfrac{\left(\varphi(t)\dot{G}(t) - \dot{\varphi}(t)G(t)\right)}{(n+2)\varphi(t)}q_1,\\
\eta^{2}\left(t,\mathbf{q}\right) = &\quad q_1^{-\frac{n+2}{2}}\left\{\cos\left(\tfrac{n+2}{2}q_2\right)\left(C_1 + C_2\int\tfrac{dt}{\varphi(t)}\right) - \sin\left(\tfrac{n+2}{2}q_2\right)\left(C_3 + C_4\int\tfrac{dt}{\varphi(t)}\right)\right\} + C_5
\end{aligned}$$

► Gauge term

$$\begin{aligned}
V\left(t,\mathbf{q}\right) \;=\; & \frac{2}{n+2}q_1^{\frac{n+2}{2}}\left(C_2\sin\left(\frac{n+2}{2}q_2\right) + C_4\cos\left(\frac{n+2}{2}q_2\right)\right) + \frac{2}{n+2}q_1^{n+2}\times\\
& \left(\dot{\varphi}\left(t\right)^2 G\left(t\right) + \varphi\left(t\right)^2\ddot{G}\left(t\right) - \dot{\varphi}\left(t\right)\dot{G}\left(t\right)\varphi\left(t\right) - \varphi\left(t\right)G\left(t\right)\ddot{\varphi}\left(t\right)\right).
\end{aligned}$$

► Constraint on function $G(t)$:

$$\ddot{z}\left(t\right) - \frac{\left(2\ddot{\varphi}\left(t\right)\varphi\left(t\right) - \dot{\varphi}\left(t\right)^2\right)}{4\varphi\left(t\right)^2}z\left(t\right) = 0. \tag{87}$$

► Symmetries for which $\xi\left(t,\mathbf{q}\right) = V\left(t,\mathbf{q}\right) = 0$ are preserved.

► Lagrangians solving the inverse problem

$$\widetilde{L}_{n,\varepsilon} = \varphi\left(t\right) \frac{q_1^n}{2} \left(\dot{q}_1^2 + q_1^2 \left(1 + 2\varepsilon\, S\left(q_2\right)\right) \dot{q}_2^2\right) - \varepsilon \frac{U\left(q_2\right)}{\varphi\left(t\right) q_1^{n+2}}. \tag{88}$$

► H not a constant of the motion, but $\widetilde{J}_1 = \varphi\left(t\right) H$.

► Second invariant

$$\widetilde{J}_2 = \varphi\left(t\right)^2 \frac{q_1^{2n+4}}{2} \left(1 + 2\varepsilon S\left(q_2\right)\right) \dot{q}_2^2 + \varepsilon U\left(q_2\right) \tag{89}$$

► $\widetilde{J}_2 \neq \varphi\left(t\right) J_2$, as

$$\varphi\left(t\right)\left(L_n + \varepsilon\, R\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)\right) - \widetilde{L}_{n,\varepsilon} = \varepsilon \frac{U\left(q_2\right)}{q_1^{n+2}} \frac{\left(\varphi^2\left(t\right) - 1\right)}{\varphi\left(t\right)} \neq 0,$$

► Symmetry-preserving perturbation and multiplication by time functions $\varphi\left(t\right)$ not interchangeable.

A PDM-VERSION OF ERMAKOV-RAY-REID SYSTEMS

▶ Generalized Ermakov-Ray-Reid systems (ERR) admitting a Hamiltonian arise as symmetry-preserving perturbations of the toscillator $\ddot{\mathbf{x}} + \omega^2(t)\,\mathbf{x} = 0$ with time-dependent frequency.

▶ In polar coordinates $q_1 = \rho$, $q_2 = \theta$, we obtain $L_0 = \frac{1}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 - \omega^2(t)\,q_1^2\right)$.

▶ Generalization to ERR-systems with pdm. Let $n \neq -2$

$$L_n = \frac{\varphi(t)\,q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 - \omega^2(t)\,q_1^2\right). \tag{90}$$

▶ Inverse problem with $\mathfrak{sl}(2,\mathbb{R})$-preservation

$$L_{n,\varepsilon} = \frac{\varphi(t)\,q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 - \omega^2(t)\,q_1^2\right) + \varepsilon\,\varphi(t)^{-1}\,q_1^{-(n+2)}\Phi\left(q_2, \varphi(t)\,q_1^{n+2}\dot{q}_2\right), \tag{91}$$

▶ with

$$\Phi\left(q_2, \varphi(t)\,q_1^{n+2}\dot{q}_2\right)$$

arbitrary function of its arguments.

▶ New variable $w = \varphi(t)\, q_1^{n+2} \dot{q}_2$: two independent constants of the motion are

$$
\begin{aligned}
J_1 &= \varphi(t)\left(\dot{q}_1 \frac{\partial L_{n,\varepsilon}}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial L_{n,\varepsilon}}{\partial \dot{q}_2} - L_{n,\varepsilon}\right) - \frac{1}{2} q_1^{n+2} \varphi^2(t)\, \omega^2(t) \\
&\quad - \int^{q_1^{n+2}} \left(\varphi(\zeta)\, \omega(\zeta)\right)^2 d\zeta, \qquad (92) \\
J_2 &= \varphi^2(t)\, q_1^{2n+4} \dot{q}_2^2 + 2\varepsilon \varphi(t)\, q_1^{n+2} \dot{q}_2 \frac{\partial \Phi}{\partial w} - 2\varepsilon \Phi(q_2, w) \qquad (93)
\end{aligned}
$$

▶ The choice $\varphi(t) = 1$ provides a pdm-version of the generalized ERR systems.

▶ We recover the two invariants of the classical ERR systems as a limit:

$$
\begin{aligned}
\lim_{n \to 0} J_1 &= \frac{1}{2}\left(\dot{q}_1^2 + q_1^2 \dot{q}_2^2\right) + \varepsilon\, \dot{q}_2 \frac{\partial \Phi}{\partial w} - \varepsilon\, q_1^{-2} \Phi(q_2, w) + \frac{1}{2} \int^{q_1^2} \omega^2(\zeta)\, d\zeta, \\
\lim_{n \to 0} J_2 &= q_1^4 \dot{q}_2^2 + 2\varepsilon\, q_1^2 \dot{q}_2 \frac{\partial \Phi}{\partial w} - 2\varepsilon\, \Phi(q_2, w). \qquad (94)
\end{aligned}
$$

A FUNCTIONAL APPROACH

- $\theta(t)$, $\rho(t)$ arbitrary functions, $\{u(t), v(t)\}$ independent solutions of

$$\ddot{z}(t) + \rho(t)\dot{z}(t) + \theta(t)z(t) = 0. \tag{95}$$

- $\xi\,(t) = C_1 u(t)^2 + C_2 u(t)v(t) + C_3 v(t)^2$ determines the general solution of

$$\dddot{\xi} + 3\dot{\rho}\ddot{\xi} + \left(\dot{\rho} + 2\rho^2 + 4\theta\right)\dot{\xi} + \left(4\rho\theta + 2\dot{\theta}\right)\xi = 0. \tag{96}$$

- Consider vector fields

$$\mathbf{X} = \xi\,(t)\,\frac{\partial}{\partial t} + \frac{1}{2}\dot{\xi}\,(t)\,q_i\frac{\partial}{\partial q_i},$$

- $\mathbf{X}_i$ $(1 \leq i \leq 3)$ is associated to $C_j = \delta_i^j$.
- Commutators .

$$\begin{aligned} &[\mathbf{X}_1, \mathbf{X}_2] = W(u,v)\mathbf{X}_1,\ [\mathbf{X}_1, \mathbf{X}_3] = 2W(u,v)\mathbf{X}_2, \\ &[\mathbf{X}_2, \mathbf{X}_3] = W(u,v)\mathbf{X}_3, \end{aligned} \tag{97}$$

$W(u,v) = u\dot{v} - \dot{u}v$ Wronskian.

- If $W(u,v) = \mathrm{cons.}$ we recover $\mathfrak{sl}(2,\mathbb{R})$.
- Define

$$V(\mathbf{q}) = \frac{1}{4}\ddot{\xi}(t)\left(q_1^2 + q_2^2\right),$$

- $\mathbf{X}_i$ are Noether point symmetries of $L(t,\mathbf{q},\dot{\mathbf{q}})$ if

$$\frac{1}{2}\frac{\partial L}{\partial \dot{q}_1}\left(\ddot{\xi}q_1 - \dot{\xi}\dot{q}_1\right) + \frac{1}{2}\frac{\partial L}{\partial \dot{q}_2}\left(\ddot{\xi}q_2 - \dot{\xi}\dot{q}_2\right) + \xi\frac{\partial L}{\partial t} + \frac{\dot{\xi}}{2}\left(q_1\frac{\partial L}{\partial q_1} + q_2\frac{\partial L}{\partial q_2}\right)$$
$$-\frac{\dddot{\xi}}{4}\left(q_1^2 + q_2^2\right) - \frac{1}{2}\ddot{\xi}\dot{q}_1 q_1 - \frac{1}{2}\ddot{\xi}\dot{q}_2 q_2 + \dot{\xi}L = 0$$

- General solution

$$L(t,\mathbf{q},\dot{\mathbf{q}}) = \frac{\left(\xi\ddot{\xi} - \dot{\xi}^2\right)}{4\xi^2}\left(q_1^2 + q_2^2\right) + \frac{\dot{\xi}\left(q_1\dot{q}_1 + \dot{q}_2 q_2\right)}{2\xi}$$
$$+\frac{1}{\xi}\Psi\left(\frac{q_1}{\sqrt{\xi}}, \frac{q_2}{\sqrt{\xi}}, \frac{\dot{\xi}q_1 - 2\dot{q}_1\xi}{\sqrt{\xi}}, \frac{\dot{\xi}q_2 - 2\dot{q}_2\xi}{\sqrt{\xi}}\right),$$

- Contains the class of oscillatory systems

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) + \frac{2\xi\ddot{(t)}\xi(t) - 1}{\xi(t)^2}\left(q_1^2 + q_2^2\right). \tag{98}$$

Perturbed PDM-systems in $N = 3$

- Starting point

$$L_n = \frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 + q_1^2\sin^2\left(q_2\right)\dot{q}_3^2\right) \tag{99}$$

- associated to the metric tensor

$$\overline{ds^2} = \frac{q_1^n}{2}\left(dq_1^2 + q_1^2 dq_2^2 + q_1^2\sin^2\left(q_2\right)dq_3^2\right). \tag{100}$$

- nonzero components of R_{ijkl}, up to symmetries

$$R_{2323} = -\frac{n\left(4+n\right)}{2}q_1^{n+2}\sin^2\left(q_2\right) \tag{101}$$

- the Riemann tensor vanishes only if $n = 0$ or $n = -4$.
- Lagrangian L_n is linearizable only for $n = 0, -4$.
- Number of Noether point symmetries is 12.

▶ $\mathcal{L}_{NS}$ Lie algebra of Noether point symmetries.

1. If $n = 0, -4$, then $\mathcal{L}_{NS}$ is isomorphic to the unextended Schrödinger algebra $S(3)$.
2. If $n = -2$, then $\mathcal{L}_{NS}$ is isomorphic to the direct sum $\mathfrak{so}(3) \oplus \mathfrak{h}_1$, where $\mathfrak{h}_1$ denotes the three dimensional Heisenberg algebra.
3. If $n \neq -4, -2, 0$, then $\mathcal{L}_{NS} \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)$ and coincides with the Levi subalgebra of $S(3)$.

▶ The value $n = -2$ must be analyzed separately, Rotational symmetry (Killing vectors) preserved, remaining Noether point symmetries

$$\xi(t,\mathbf{q}) = C_1, \qquad \eta^1(t,\mathbf{q}) = C_2 t\, q_1 + C_3 q_1,$$
$$\eta^2(t,\mathbf{q}) = C_4 \sin q_3 + C_5 \cos q_3, \quad \eta^3(t,\mathbf{q}) = C_4 \cos q_3 \cot q_2 - C_5 \sin q_3 \cot q_2 + C_6.$$

▶ C_1, C_2 and C_3 generate the Heisenberg algebra $\mathfrak{h}_1$.

- The most general perturbation $\widetilde{L}_n = L_n + \varepsilon\, R\left(t, \mathbf{q}, \dot{\mathbf{q}}\right)$ preserving $\mathfrak{sl}\left(2, \mathbb{R}\right)$ is

$$R\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) = \Phi\left(q_2, q_3, \dot{q}_2 q_1^{n+2}, \dot{q}_3 q_1^{n+2}\right) q_1^{-(n+2)}. \tag{102}$$

- Imposing the symmetry condition leads to

$$\begin{aligned} \frac{\partial R}{\partial t} &= 0 \\ \frac{t^2\left(n+2\right)}{2}\frac{\partial R}{\partial t} + t\, q_1 \frac{\partial R}{\partial q_1} + \left(q_1 - t\dot{q}_1\left(n+1\right)\right)\frac{\partial R}{\partial \dot{q}_1} - t\left(n+2\right) + \\ \left(\dot{q}_2 \frac{\partial R}{\partial \dot{q}_2} + \dot{q}_3 \frac{\partial R}{\partial \dot{q}_3}\right) + t\left(n+2\right) R\left(t, \mathbf{q}, \dot{\mathbf{q}}\right) &= 0. \end{aligned}$$

- simplified to

$$q_1 \frac{\partial R}{\partial q_1} - \left(n+2\right)\left(\dot{q}_2 \frac{\partial R}{\partial \dot{q}_2} + \dot{q}_3 \frac{\partial R}{\partial \dot{q}_3}\right) + \left(n+2\right) R\left(\mathbf{q}, \dot{\mathbf{q}}\right) = 0,\ \ q_1 \frac{\partial R}{\partial \dot{q}_1} = 0. \tag{103}$$

▶ For $\widetilde{L}_{n,\varepsilon}$ with $\varphi_4(q_2, q_3) = 0$. If

$$\frac{\partial \varphi_i}{\partial q_3} \neq 0$$

no symmetries corresponding to Killing vectors.

▶ To ensure enough independent invariants, larger subgroup required.

EXAMPLE:

$$\widetilde{L}_{n,\varepsilon} = \frac{q_1^n}{2}\left\{\dot{q}_1^2 + q_1^2\left((1+\varepsilon\varphi_1(q_2))\,\dot{q}_2^2 + 2\varepsilon\varphi_2(q_2)\,\dot{q}_2\dot{q}_3 + \left(\sin^2(q_2) + \varepsilon\varphi_3(q_2)\right)\dot{q}_3^2\right)\right\} + \frac{\varepsilon\varphi_4(q_2)}{q_1^{n+2}}.$$

▶ Admitted invariants

$$\begin{aligned} J_1 &= \frac{q_1^n}{2}\left\{\dot{q}_1^2 + q_1^2\left((1+2\varepsilon\varphi_1(q_2))\,\dot{q}_2^2 + 4\varepsilon\varphi_2(q_2)\,\dot{q}_2\dot{q}_3 + \left(\sin^2(q_2) + 2\varepsilon\varphi_3(q_2)\right)\dot{q}_3^2\right)\right\} - \frac{\varepsilon\varphi_4(q_2)}{q_1^{n+2}}, \\ J_2 &= q_1^{n+2}\left(2\varepsilon\varphi_2(q_2)\,\dot{q}_2 + \left(\sin^2 q_2 + 2\varepsilon\,\varphi_2(q_2)\,\dot{q}_3\right)\right), \\ J_3 &= q_1^{2n+4}\left((1+2\varepsilon\varphi_1(q_2))\,\dot{q}_2^2 + 4\varepsilon\varphi_2(q_2)\,\dot{q}_2\dot{q}_3 + \left(\sin^2(q_2) + 2\varepsilon\varphi_3(q_2)\right)\dot{q}_3^2\right) - 2\varepsilon\varphi_4(q_2). \end{aligned}$$

▶ Higher orders in $\dot{\mathbf{q}}$ can be obtained by dynamical symmetries.

PERTURBATIONS ASSOCIATED TO FLAT METRICS

- For any $n \neq -2$,

$$L_n = \frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2 \dot{q}_2^2 + q_1^2 \sin^2(q_2)\, \dot{q}_3^2\right) \tag{104}$$

admits perturbations with $\mathcal{L}_{NS} \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ and flat metric:

$$\widetilde{R}_{ijkl} = 0,\ 1 \leq i,j,k,l \leq 3.$$

- As special solution among perturbations we find

$$\widetilde{L}_{n,\varepsilon} = \frac{q_1^n}{2}\left\{\dot{q}_1^2 + q_1^2\, \left(\dot{q}_2^2 + 2\varepsilon\varphi_2(q_2)\,\dot{q}_2\dot{q}_3 + \sin^2(q_2)\,\dot{q}_3^2\right)\right\}, \tag{105}$$

- Curvature tensor

$$\begin{aligned}\widetilde{R}_{2323} \quad &= \frac{n(4+n)\sin^4 q_2 + \varepsilon^2\left(n(4+n)\cos(2q_2) - (2+n)^2\right)\varphi_2^2(q_2) + \varepsilon^4(2+n)^4\,\varphi_2^4(q_2)}{q_1^{n+2}\left(\varepsilon\varphi_2^2(q_2) - \sin^2 q_2\right)^2}\\ &+ \frac{2\varepsilon^2\sin(2q_2)\,\varphi_2(q_2)\,\frac{d\varphi_2}{dq_2}}{q_1^{n+2}\left(\varepsilon\varphi_2^2(q_2) - \sin^2 q_2\right)^2}.\end{aligned} \tag{106}$$

▶ Parameterized nonlinear first order differential equation for $\varphi_2(q_2)$:[18]

$$\frac{n(4+n)\sin^4 q_2}{\varphi_2(q_2)}+\varepsilon^2\left(n(4+n)\cos(2q_2)-(2+n)^2\right)\varphi_2(q_2)+\varepsilon^4(2+n)^4\varphi_2^3(q_2)+2\varepsilon^2\sin(2q_2)\frac{d\varphi_2}{dq_2}=0.$$

▶ Solution

$$\varphi_2(q_2)=\varepsilon^{-1}\frac{\sqrt{C_1+n(4+n)\cos^2 q_2}}{\sqrt{(2+n)^2\cot^2 q_2+C_1\csc^2 q_2}},\ C_1\in\mathbb{R}. \tag{107}$$

▶ Stronger assertion: If

$$R_{ij}=0\Rightarrow R_{ijkl}=0.$$

▶ Solutions of perturbations correspond to geodesic lines of flat metrics.

[18] Appell M P 1889 *Journ. de Math.* **5** 361.

PDM-SYSTEMS WITH DISSIPATIVE TERMS

- Lagrangian of Kepler system

$$L = \frac{1}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 + q_1^2\sin^2(q_2)\,\dot{q}_3^2 + \frac{\alpha}{q_1}\right), \tag{108}$$

- Obvious generalization

$$L_n = \frac{q_1^n}{2}\left(\dot{q}_1^2 + q_1^2\dot{q}_2^2 + q_1^2\sin^2(q_2)\,\dot{q}_3^2 + \frac{\alpha}{q_1}\right). \tag{109}$$

- Restricting to Noether point symmetries

1. If $n \neq 1$, then $\dim \mathcal{L}_{NS} = 4$ with $\mathcal{L}_{NS} \simeq \mathbb{R} \oplus \mathfrak{so}(3)$.
2. If $n = 1$, then $\dim \mathcal{L}_{NS} = 6$ with $\mathcal{L}_{NS} \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3)$.

- Are there superintegrable solutions?

- Special value $n = 1$

$$\widetilde{L}_1 = \frac{q_1}{2}\left(\dot{q}_1^2 + q_1^2\,(1+2\gamma)\left(\dot{q}_2^2 + \sin^2(q_2)\,\dot{q}_3^2\right) + \frac{\alpha}{q_1}\right) + \frac{\beta}{q_1^3}.$$

- $\mathfrak{so}(3)$-preserving perturbation term

$$R\,(t,\mathbf{q},\dot{\mathbf{q}}) = \Phi\left(t, q_1, \dot{q}_1, \dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right). \tag{110}$$

- $\mathcal{L}_{NS}$-preserving perturbation term

$$\Phi\,(q_1, w) = \frac{1}{q_1^3}\Psi\left(q_1^6 w\right) = \frac{1}{q_1^3}\Psi\left(q_1^6\left(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right)\right). \tag{111}$$

- Quadratic in $\dot{\mathbf{q}}$

$$R\,(t,\mathbf{q},\dot{\mathbf{q}}) = C_1\, q_1^3\left(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right) + \frac{\beta}{q_1^3}, \tag{112}$$

- Constants of the motion:

$$
\begin{aligned}
J_1 &= \frac{q_1}{2}\left\{\dot{q}_1^2 + q_1^2\,(1+2\gamma)\left(\dot{q}_2^2 + \sin^2(q_2)\,\dot{q}_3^2\right) - \frac{\alpha}{q_1}\right\} - \frac{\beta}{q_1^3} \\
J_2 &= q_1^3 \sin^2 q_2 \dot{q}_3, \\
J_3 &= q_1^6\left(\dot{q}_2^2 + \sin^2 q_2 \dot{q}_3^2\right), \\
J_4 &= 2\sin q_3\,\dot{q}_2 + \cos q_3 \sin(2q_2)\;\dot{q}_3,
\end{aligned}
$$

- No solutions being maximally superintegrable.
- Other generalizations

$$
\widetilde{L}_1 = \Psi(\mathbf{q},\dot{\mathbf{q}})\left(\dot{q}_1^2 + q_1^2\,(1+2\gamma)\left(\dot{q}_2^2 + \sin^2(q_2)\,\dot{q}_3^2\right) + \frac{\alpha}{q_1}\right) + U(\mathbf{q})
$$

possible, however not being maximally superintegrable.

APPROACH VIA REALIZATIONS OF VF

- Generic $\mathfrak{sl}(2,\mathbb{R})$ realization

$$X_1 = t^2\frac{\partial}{\partial t} + 2\alpha q_1^k t\frac{\partial}{\partial q_1},\ X_2 = t\frac{\partial}{\partial t} + \alpha q_1^k\frac{\partial}{\partial q_1},\ X_3 = \frac{\partial}{\partial t}, \tag{113}$$

- NPS of Lagrangians

$$L = \frac{1}{2}\exp\left(\frac{q_1^{1-k}}{\alpha(1-k)}\right)\left(q_1^{-2k}\varphi_1(\mathbf{q}')\,\dot{q}_1^2 + 2\alpha q_1^{-k}\sum_{j=1}^{N}\frac{\partial\varphi_1}{\partial q_j}\dot{q}_1\dot{q}_j + \sum_{j,k=2}^{N}\varphi_{ij}(\mathbf{q}')\,\dot{q}_j\dot{q}_k\right) - \exp\left(\frac{-q_1^{1-k}}{\alpha(1-k)}\right)U(\mathbf{q}'),$$

$$\mathbf{q'} = (q_2,\cdots,q_N), \quad \varphi_1(\mathbf{q}'),\varphi_{ij}(\mathbf{q}'),U(\mathbf{q}')$$

- Gauge term

$$V(t,\mathbf{q}) = 2\alpha^2\varphi_1(\mathbf{q}')\exp\left(\frac{q_1^{1-k}}{\alpha(1-k)}\right). \tag{114}$$

- $N = 2, U(q_2) = 0$ metric

$$\overline{ds^2} = \frac{1}{2}\exp\left(\frac{q_1^{1-k}}{\alpha(1-k)}\right)\left(q_1^{-2k}\varphi_1(q_2)\,\dot{q}_1^2 + 2\alpha q_1^{-k}\frac{d\varphi_1}{\partial q_2}\dot{q}_1\dot{q}_2 + \varphi_{22}(q_2)\,\dot{q}_2^2\right),$$

$$\varphi_{12}(q_2)\,\varphi_1(q_2) - 4\left(\frac{d\varphi_1}{dq_2}\right)^2 \neq 0.$$

- Riemann tensor

$$R_{1212} = -\frac{1}{8}\exp\left(\frac{q_1^{1-k}}{\alpha(1-k)}\right)\frac{\left(2\frac{d^2\varphi_1}{dq_2^2}\varphi_1(q_2) - \frac{d\varphi_1}{dq_2}^2\right)\varphi_{12}(q_2) - \varphi_1(q_2)\frac{d\varphi_1}{dq_2}\frac{d\varphi_{12}}{dq_2}}{2\alpha^2\frac{d\varphi_1}{dq_2}^2 - \varphi_1(q_2)\,\varphi_{12}(q_2)},$$

- Vanishes only if

$$\varphi_{12}(q_2) = \lambda\left(\frac{d\varphi_1}{dq_2}\right)^2/\varphi_1(q_2).$$

AN n-DIMENSIONAL EXTRAPOLATION

- Metric

$$ds^2 = \frac{1}{(q^n)^2}\left(dq^1 \otimes dx^1 + \cdots + dq^n \otimes dq^n\right) \tag{115}$$

- Conformal group $SO(1,n)$ as isometry group.
- $\mathcal{L}_{PS} \simeq \mathfrak{so}(1,n) \oplus \mathfrak{r}_2$ [$\mathfrak{r}_2$ the 2-dimensional affine Lie algebra]
- The vector field $\mathbf{Y} = t\frac{\partial}{\partial t}$ leads to

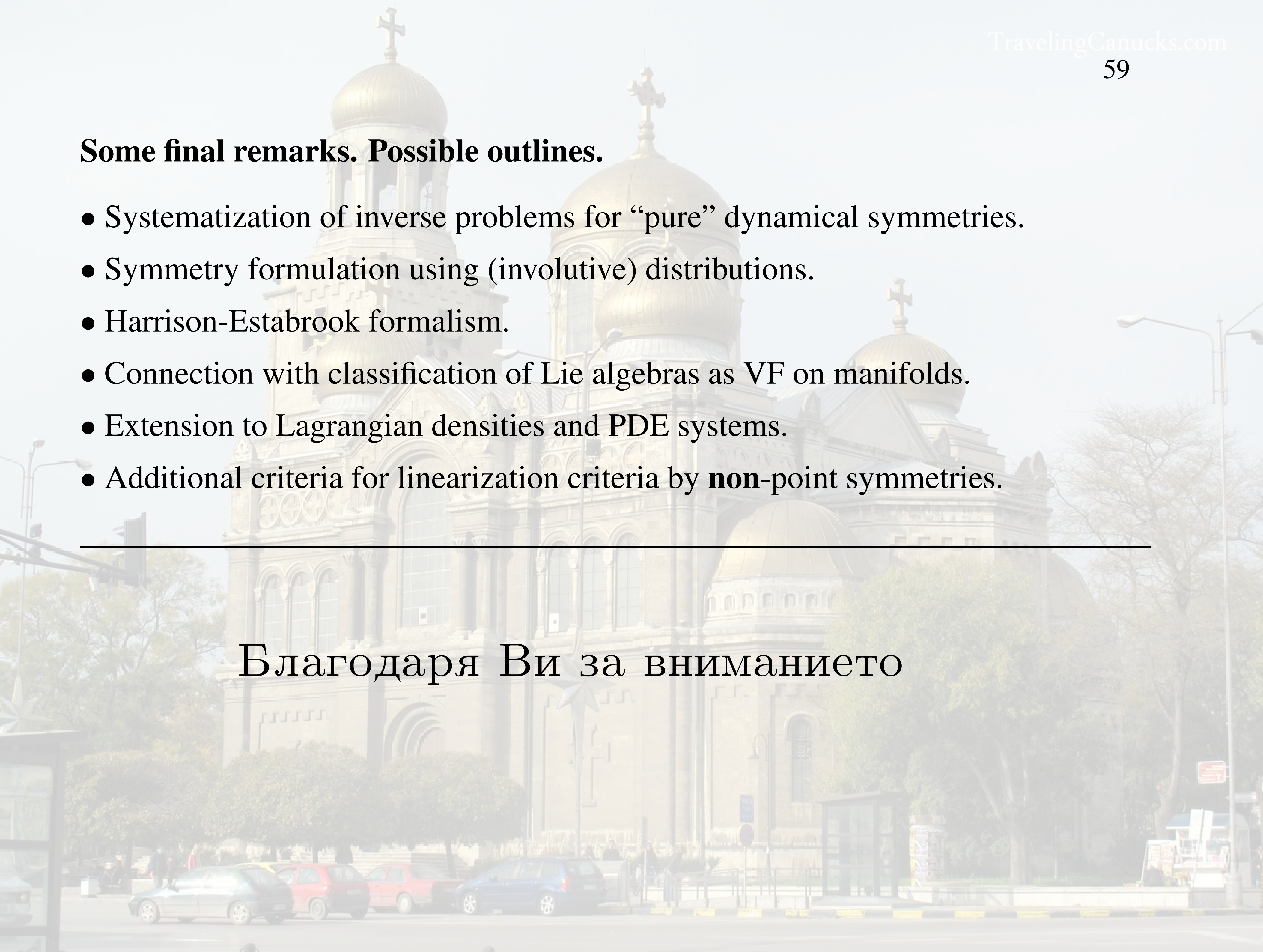
$$-\frac{1}{2q_n^2}\sum_{k=1}^{n}\left(\dot{q}_k^2 + \dot{q}_k\frac{\partial V}{\partial q_k}\right) - \frac{\partial V}{\partial t}, \tag{116}$$

no gauge term $V(t,\mathbf{q})$ possible.

- $\mathcal{L}_{NS} \simeq \mathfrak{so}(1,n) \oplus \mathbb{R}$
- System $(L = L_0 - U(t,\mathbf{q}))$

$$\ddot{q}_k = \frac{2\dot{q}_k\dot{q}_n}{q_n} - \frac{q_n^2}{2}\frac{\partial U}{\partial q_k},\ 1 \leq k \leq n-1;\ \ddot{q}_n = \frac{\dot{q}_n^2 - \dot{q}_1^2 - \cdots - \dot{q}_{n-1}^2}{q_n} - \frac{q_n^2}{2}\frac{\partial U}{\partial q_n}$$

- $\mathcal{L}_{NS} \subset \mathfrak{so}(1,n) \oplus \mathbb{R}$ is a proper subalgebra.



Some final remarks. Possible outlines.

- Systematization of inverse problems for “pure” dynamical symmetries.
- Symmetry formulation using (involutive) distributions.
- Harrison-Estabrook formalism.
- Connection with classification of Lie algebras as VF on manifolds.
- Extension to Lagrangian densities and PDE systems.
- Additional criteria for linearization criteria by **non**-point symmetries.

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