Quantum Localisation on the Circle

XXth International Conference on Geometry, Integrability and Quantization

Rodrigo Fresneda (UFABC - São Paulo, Brasil)

Varna, 6th of June of 2018

If $\psi(\alpha)$ is the $2\pi$-periodic wave function on the circle, the quantum angle $\hat{\alpha}$ cannot be a multiplication operator, $\hat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.
If $\psi(\alpha)$ is the $2\pi$-periodic wave function on the circle, the quantum angle $\hat{\alpha}$ cannot be a multiplication operator, $\hat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.

Except if $\hat{\alpha}$ stands for the $2\pi$-periodic discontinuous angle function,

$$ (\hat{\alpha}\psi)(\alpha) := \left( \alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor \right) \psi(\alpha). \quad (1) $$
If $\psi(\alpha)$ is the $2\pi$-periodic wave function on the circle, the quantum angle $\hat{\alpha}$ cannot be a multiplication operator, $\hat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.

Except if $\hat{\alpha}$ stands for the $2\pi$-periodic discontinuous angle function,

$$(\hat{\alpha}\psi)(\alpha) := \left(\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor \right) \psi(\alpha). \quad (1)$$

However, for $\hat{\alpha}$ Self-Adjoint (SA), $\text{spec}(\hat{\alpha}) \subset [0, 2\pi]$, the CCR $[\hat{\alpha}, \hat{p}_\alpha] = i\hbar I$ does not hold for SA quantum angular momentum $\hat{p}_\alpha = -i\hbar \frac{\partial}{\partial \alpha}$. 
If $\psi(\alpha)$ is the $2\pi$-periodic wave function on the circle, the quantum angle $\hat{\alpha}$ cannot be a multiplication operator, $\hat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.

Except if $\hat{\alpha}$ stands for the $2\pi$-periodic discontinuous angle function,

$$(\hat{\alpha}\psi)(\alpha) := (\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor) \psi(\alpha). \quad (1)$$

However, for $\hat{\alpha}$ Self-Adjoint (SA), $\text{spec}(\hat{\alpha}) \subset [0, 2\pi]$, the CCR $[\hat{\alpha}, \hat{p}_\alpha] = i\hbar I$ does not hold for SA quantum angular momentum $\hat{p}_\alpha = -i\hbar \frac{\partial}{\partial \alpha}$.

Instead, one has

$$[\hat{\alpha}, \hat{p}_\alpha] = i\hbar I \left[ 1 - 2\pi \sum_n \delta(\alpha - 2n\pi) \right]. \quad (2)$$
This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").
• This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").

• Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.
• This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").

• Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.

• We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).
This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").

Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.

We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).

Our approach is group theoretical, based on the unitary irreducible representations of the (special) Euclidean group $E(2) = \mathbb{R}^2 \rtimes SO(2)$ (see also S. De Bièvre, Coherent states over symplectic homogeneous spaces).
This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").

Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.

We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, Coherent States, Wavelets and their Generalizations).

Our approach is group theoretical, based on the unitary irreducible representations of the (special) Euclidean group $E(2) = \mathbb{R}^2 \rtimes SO(2)$ (see also S. De Bièvre, Coherent states over symplectic homogeneous spaces).

One of our aims is to build acceptable angle operators from the classical angle function through a consistent and manageable quantisation procedure.
Let $G$ be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of $G$ in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_{G} \rho(g) \, d\mu(g), \quad \rho(g) := U(g) \rho U^\dagger(g)$$
Covariant integral quantisation - general scheme

Let $G$ be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of $G$ in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g) \, d\mu(g), \quad \rho(g) := U(g) \rho U^\dagger(g)$$

Then, $R = c_\rho I$, since $U(g_0)RU^\dagger(g_0) = \int_G \rho(g_0g) \, d\mu(g) = R$
Let $G$ be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of $G$ in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$ R := \int_G \rho(g) \, d\mu(g), \quad \rho(g) := U(g) \rho U^\dagger(g) $$

Then, $R = c_\rho I$, since

$$ U(g_0) R U^\dagger(g_0) = \int_G \rho(g_0 g) \, d\mu(g) = R $$

That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$ \int_G \rho(g) \frac{d\mu(g)}{c_\rho} = I, \quad c_\rho = \int_G \text{tr}(\rho_0 \rho(g)) \, d\mu(g) $$
Let $G$ be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of $G$ in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g) \, d\mu(g), \quad \rho(g) := U(g) \rho U^\dagger(g)$$

Then, $R = c_\rho I$, since $U(g_0) R U^\dagger(g_0) = \int_G \rho(g_0 g) \, d\mu(g) = R$.

That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$\int_G \rho(g) \frac{d\mu(g)}{c_\rho} = I, \quad c_\rho = \int_G \text{tr}(\rho_0 \rho(g)) \, d\mu(g)$$

This allows an integral quantisation of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g) \, f(g) \frac{d\mu(g)}{c_\rho},$$
Let $G$ be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of $G$ in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g) \, d\mu(g), \quad \rho(g) := U(g) \rho U^\dagger(g)$$

Then, $R = c_\rho I$, since $U(g_0) R U^\dagger(g_0) = \int_G \rho(g_0 g) \, d\mu(g) = R$

That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$\int_G \rho(g) \frac{d\mu(g)}{c_\rho} = I, \quad c_\rho = \int_G \text{tr}(\rho_0 \rho(g)) \, d\mu(g)$$

This allows an integral quantisation of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g) f(g) \frac{d\mu(g)}{c_\rho},$$

which is covariant in the sense that

$$U(g) A_f U^\dagger(g') = A_{U(g)f}, \quad (U(g)f)(g') = f(g^{-1}g')$$
We consider the quantisation of functions on a homogeneous space $X$, the left coset manifold $X \sim G/H$ for the action of a Lie Group $G$, where the closed subgroup $H$ is the stabilizer of some point of $X$. The interesting case is when $X$ is a symplectic manifold (e.g., co-adjoint orbit of $G$) and can be viewed as the phase space for the dynamics. Given a quasi-invariant measure $\nu$ on $X$, one has for a global Borel section $\sigma$: $X \rightarrow G$ a unique quasi-invariant measure $\nu_\sigma(x)$.

Let $U$ be a square-integrable UIR, and $\rho_0$ a density operator such that $c\rho_0 := \int_X \text{tr}(\rho_0 \rho_\sigma(x)) d\nu_\sigma(x) < \infty$ with $\rho_\sigma(x) := U(\sigma(x)) \rho U(\sigma(x))^\dagger$. Rodrigo Fresneda (UFABC - São Paulo, Brasil)
We consider the quantisation of functions on a homogeneous space \( X \), the left coset manifold \( X \sim G/H \) for the action of a Lie Group \( G \), where the closed subgroup \( H \) is the stabilizer of some point of \( X \).

The interesting case is when \( X \) is a symplectic manifold (e.g., co-adjoint orbit of \( G \)) and can be viewed as the phase space for the dynamics.
We consider the quantisation of functions on a homogeneous space \( X \), the left coset manifold \( X \sim G/H \) for the action of a Lie Group \( G \), where the closed subgroup \( H \) is the stabilizer of some point of \( X \).

The interesting case is when \( X \) is a symplectic manifold (e.g., co-adjoint orbit of \( G \)) and can be viewed as the phase space for the dynamics.

Given a quasi-invariant measure \( \nu \) on \( X \), one has for a global Borel section \( \sigma : X \to G \) a unique quasi-invariant measure \( \nu_\sigma(X) \).
We consider the quantisation of functions on a homogeneous space $X$, the left coset manifold $X \sim G/H$ for the action of a Lie Group $G$, where the closed subgroup $H$ is the stabilizer of some point of $X$.

The interesting case is when $X$ is a symplectic manifold (e.g., co-adjoint orbit of $G$) and can be viewed as the phase space for the dynamics.

Given a quasi-invariant measure $\nu$ on $X$, one has for a global Borel section $\sigma : X \rightarrow G$ a unique quasi-invariant measure $\nu_\sigma(x)$.

Let $U$ be a square-integrable UIR, and $\rho_0$ a density operator such that $c_\rho := \int_X \text{tr} (\rho_0 \rho_\sigma(x)) \, d\nu_\sigma(x) < \infty$ with $\rho_\sigma(x) := U(\sigma(x)) \rho U(\sigma(x))^\dagger$. 

One has the resolution of the identity

\[ I = \frac{1}{c_\rho} \int_X \rho_\sigma(x) \, d\nu_\sigma(x). \]
- One has the resolution of the identity

\[ I = \frac{1}{c_\rho} \int_X \rho_\sigma(x) \, d\nu_\sigma(x) . \]

- We then define the quantisation of functions on \( X \) as the linear map

\[ f \mapsto A_\sigma^f = \frac{1}{c_\rho} \int_X f(x) \rho_\sigma(x) \, d\nu_\sigma(x) . \]
One has the resolution of the identity

\[ I = \frac{1}{c_\rho} \int_X \rho_\sigma(x) \, d\nu_\sigma(x). \]

We then define the quantisation of functions on \( X \) as the linear map

\[ f \mapsto A^\sigma_f = \frac{1}{c_\rho} \int_X f(x) \rho_\sigma(x) \, d\nu_\sigma(x). \]

Covariance holds in the sense \( U(g) A^\sigma_f U(g)^\dagger = A^{\sigma g}_{U_l(g)f} \), where \( \sigma_g(x) = g \sigma(g^{-1}x) \) with \( U_l(g)f(x) = f(g^{-1}x) \).

For \( \rho = |\eta\rangle \langle \eta| \), we are working with CS quantisation, where the CS’s are defined as \( |\eta_x\rangle := |U(\sigma_g(x))\eta\rangle \).
Coherent states for semi-direct product groups

Let $V$, $\text{dim} V = n$, $S \leq \text{GL}(V)$ and $G = V \rtimes S$
Let $V$, $\dim V = n$, $S \leq GL(V)$ and $G = V \rtimes S$

Given $k_0 \in V^*$, one can show that

$$H_0 = \{ g \in G | (k_0, 0) = \text{Ad}^g_{\#}(k_0, 0) \} = N_0 \rtimes S_0$$

for $(k_0, 0) \in g^*$
• Let $V$, $\dim V = n$, $S \leq GL(V)$ and $G = V \rtimes S$
• Given $k_0 \in V^*$, one can show that

$$H_0 = \{g \in G | (k_0, 0) = \text{Ad}^\#_g (k_0, 0)\} = N_0 \rtimes S_0$$

for $(k_0, 0) \in \mathfrak{g}^*$

• Furthermore, $X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^*\mathcal{O}^*$, $V_0 = T^*\mathcal{O}^*$, is a symplectic manifold with symplectic measure $d\mu(p, q)$ which allows the construction of a section $V_0 \times \mathcal{O}^* \ni (p, q) \mapsto \sigma(p, q) \in G$
Coherent states for semi-direct product groups

- Let $V$, $\dim V = n$, $S \leq GL(V)$ and $G = V \rtimes S$
- Given $k_0 \in V^*$, one can show that

$$H_0 = \{ g \in G | (k_0, 0) = \operatorname{Ad}_g^#(k_0, 0) \} = N_0 \rtimes S_0$$

for $(k_0, 0) \in \mathfrak{g}^*$
- Furthermore, $X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^* \mathcal{O}^*$, $V_0 = T_{k_0}^* \mathcal{O}^*$, is a symplectic manifold with symplectic measure $d\mu(p, q)$ which allows the construction of a section

$$V_0 \times \mathcal{O}^* \ni (p, q) \mapsto \sigma(p, q) \in G$$

- Finally, given a UIR $\chi$ of $V$ and a UIR $L$ of $S$, one can construct an irreducible representation $(v, s) \mapsto \chi^L U(v, s)$ of $G$ induced by the representation $\chi \otimes L$ of $V \rtimes S_0$. 
Let \( V, \dim V = n, S \leq GL(V) \) and \( G = V \rtimes S \)

Given \( k_0 \in V^* \), one can show that

\[
H_0 = \{ g \in G | (k_0, 0) = \text{Ad}_g^\#(k_0, 0) \} = N_0 \rtimes S_0
\]

for \( (k_0, 0) \in \mathfrak{g}^* \)

Furthermore, \( X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^* \mathcal{O}^* \), \( V_0 = T_{k_0}^* \mathcal{O}^* \), is a symplectic manifold with symplectic measure \( d\mu(p, q) \) which allows the construction of a section

\[
V_0 \times \mathcal{O}^* \ni (p, q) \mapsto \sigma(p, q) \in G
\]

Finally, given a UIR \( \chi \) of \( V \) and a UIR \( L \) of \( S \), one can construct an irreducible representation \((v, s) \mapsto \chi^L U(v, s)\) of \( G \) induced by the representation \( \chi \otimes L \) of \( V \rtimes S_0 \).

Given \( \eta \in \mathcal{H} = L^2(\mathcal{O}^*, d\nu) \), one constructs a family \( \eta_{p,q} : \eta_{p,q}(k) = (\chi^L U(\sigma(p, q))\eta)(k) \)
If one can prove
\[ \int_{V_0 \times O^*} d\mu(p, q) \langle \phi | \eta_{p, q} \rangle \mathcal{H} \langle \eta_{p, q} | \psi \rangle \mathcal{H} = c_\eta \langle \phi | \psi \rangle \]
where \( c_\eta \) is a constant with \( 0 < c_\eta < \infty \).
If one can prove
\[
\int_{V_0 \times O^*} d\mu(p, q) \langle \phi | \eta_{p,q} \rangle_{\mathbf{H}} \langle \eta_{p,q} | \psi \rangle_{\mathbf{H}} = c_\eta \langle \phi | \psi \rangle
\]
where \( \phi, \psi : O^* \rightarrow \mathbb{C} \) and \( 0 < c_\eta < \infty \)
we obtain the resolution of the identity
\[
\frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p,q} \rangle \langle \eta_{p,q} | = I
\]
If one can prove
\[
\int_{V_0 \times \mathcal{O}^*} d\mu(p, q) \langle \phi | \eta_{p,q} \rangle_{\mathcal{H}} \langle \eta_{p,q} | \psi \rangle_{\mathcal{H}} = c_\eta \langle \phi | \psi \rangle
\]
where
\[
\phi, \psi : \mathcal{O}^* \to \mathbb{C}
\]
and \(0 < c_\eta < \infty\), we obtain the resolution of the identity
\[
\frac{1}{c_\eta} \int_{V_0 \times \mathcal{O}^*} d\mu(p, q) | \eta_{p,q} \rangle \langle \eta_{p,q} | = I
\]
CS quantisation maps the classical function
\[
f(p, q) \in V_0 \times \mathcal{O}^*
\]
to the operator on \(\mathcal{H}\)
\[
A_f = \frac{1}{c_\eta} \int_{V_0 \times \mathcal{O}^*} d\mu(p, q) | \eta_{p,q} \rangle \langle \eta_{p,q} | f(p, q)
\]
If one can prove
\[ \int_{V_0 \times O^*} d\mu(p, q) \langle \phi | \eta_{p,q} \rangle \mathcal{H} \langle \eta_{p,q} | \psi \rangle \mathcal{H} = c_\eta \langle \phi | \psi \rangle \]
where
\[ \phi, \psi : O^* \rightarrow \mathbb{C} \]
and \(0 < c_\eta < \infty\)
we obtain the resolution of the identity
\[ \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p,q} \rangle \langle \eta_{p,q} | = I \]

CS quantisation maps the classical function
\[ f(p, q) \in V_0 \times O^* \]
to the operator on \(\mathcal{H}\)
\[ A_f = \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p,q} \rangle \langle \eta_{p,q} | f(p, q) \]
The quantisation is covariant \(\chi^L U(g) A_f \chi^L U(g)^\dagger = A_{\sigma g}^\dagger U_f(g) \chi^L \)
\[ A_{\sigma g}^\dagger = \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p,q}^{\sigma g} \rangle \langle \eta_{p,q}^{\sigma g} | f(p, q) \]
with
\[ | \eta_{p,q}^{\sigma g} \rangle = \chi^L U(g \sigma(g^{-1}(p, q))) | \eta \rangle \]
If one can prove
\[ \int_{V_0 \times O^*} d\mu(p, q) \langle \phi | \eta_{p, q} \rangle_{\mathcal{H}} \langle \eta_{p, q} | \psi \rangle_{\mathcal{H}} = c_\eta \langle \phi | \psi \rangle \]
where \( \phi, \psi : O^* \rightarrow \mathbb{C} \) and \( 0 < c_\eta < \infty \)
we obtain the resolution of the identity
\[ \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p, q} \rangle \langle \eta_{p, q} | = I \]
CS quantisation maps the classical function
\( f(p, q) \in V_0 \times O^* \) to the operator on \( \mathcal{H} \)
\[ A_f = \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p, q} \rangle \langle \eta_{p, q} | f(p, q) \]
The quantisation is covariant \( \chi^L U(g) A_f \chi^L U(g)^\dagger = A_{\chi^L U(g)}^\sigma \),
\[ A_f = A_{\chi^L U(g)}^\sigma := \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p, q) | \eta_{p, q}^\sigma \rangle \langle \eta_{p, q}^\sigma | f(p, q) \],
with \( | \eta_{p, q}^\sigma \rangle = \chi^L U(g \sigma(g^{-1}(p, q))) | \eta \rangle \)
The semiclassical portrait of the operator \( A_f \) is defined as
\[ \tilde{f}(p, q) = \frac{1}{c_\eta} \int_{V_0 \times O^*} d\mu(p', q') f(p', q') | \eta_{p', q'} \rangle \langle \eta_{p, q} |^2 \]
Now $G = E(2)$, where $V = \mathbb{R}^2$ and $S = SO(2)$, so
$E(2) = \mathbb{R}^2 \times SO(2) = \{(r, \theta), r \in \mathbb{R}^2, \theta \in [0, 2\pi)\}$, with
composition $(r, \theta)(r', \theta') = (r + R(\theta)r', \theta + \theta')$. 
Now $G = E(2)$, where $V = \mathbb{R}^2$ and $S = SO(2)$, so $E(2) = \mathbb{R}^2 \rtimes SO(2) = \{(r, \theta), \ r \in \mathbb{R}^2, \ \theta \in [0, 2\pi)\}$, with composition $(r, \theta)(r', \theta') = (r + R(\theta)r', \theta + \theta')$.

$V^* = \mathbb{R}^2$, $O^* = \{k = R(\theta)k_0 \in \mathbb{R}^2 | R(\theta) \in SO(2)\} \simeq S^1$.
Now $G = E(2)$, where $V = \mathbb{R}^2$ and $S = SO(2)$, so $E(2) = \mathbb{R}^2 \rtimes SO(2) = \{(r, \theta), \ r \in \mathbb{R}^2, \ \theta \in [0, 2\pi)\}$, with composition $(r, \theta)(r', \theta') = (r + R(\theta)r', \theta + \theta')$.

$V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{k = R(\theta)k_0 \in \mathbb{R}^2 | R(\theta) \in SO(2)\} \simeq S^1$

The stabilizer under the coadjoint action $Ad^\#_{E(2)}$ is $H_0 = \{(x, 0) \in E(2) | \hat{c} \cdot x = 0, \ \hat{c} \in \mathbb{R}^2, \ ||\hat{c}|| = 1, \ fixed\}$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)
The Euclidean group $E(2)$

- Now $G = E(2)$, where $V = \mathbb{R}^2$ and $S = SO(2)$, so
  $E(2) = \mathbb{R}^2 \rtimes SO(2) = \{(r, \theta), \ r \in \mathbb{R}^2, \ \theta \in [0, 2\pi)\}$, with composition $(r, \theta)(r', \theta') = (r + R(\theta)r', \theta + \theta')$.

- $V^* = \mathbb{R}^2$, $O^* = \{k = R(\theta)k_0 \in \mathbb{R}^2 | R(\theta) \in SO(2)\} \simeq S^1$

- The stabilizer under the coadjoint action $\text{Ad}_{E(2)}^\#$ is $H_0 = \{(x, 0) \in E(2) | \hat{c} \cdot x = 0, \ \hat{c} \in \mathbb{R}^2, \ ||\hat{c}|| = 1, \ \text{fixed}\}$.

- The classical phase space $X \equiv T^*S^1 \simeq (\mathbb{R}^2 \rtimes SO(2))/H_0 \simeq \mathbb{R} \times S^1$ carries coordinates $(p, q)$ and has symplectic measure $dp \wedge dq$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil) 
Quantum Localisation on the Circle
Now $G = E(2)$, where $V = \mathbb{R}^2$ and $S = SO(2)$, so $E(2) = \mathbb{R}^2 \rtimes SO(2) = \{(r, \theta), r \in \mathbb{R}^2, \theta \in [0, 2\pi)\}$, with composition $(r, \theta)(r', \theta') = (r + R(\theta)r', \theta + \theta')$.

$V^* = \mathbb{R}^2$, $O^* = \{k = R(\theta)k_0 \in \mathbb{R}^2 | R(\theta) \in SO(2)\} \simeq S^1$

The stabilizer under the coadjoint action $Ad^\#_{E(2)}$ is $H_0 = \{(x, 0) \in E(2) | \hat{c} \cdot x = 0, \hat{c} \in \mathbb{R}^2, \|\hat{c}\| = 1, \text{fixed}\}$.

The classical phase space $X \equiv T^*S^1 \simeq (\mathbb{R}^2 \rtimes SO(2))/H_0 \simeq \mathbb{R} \times S^1$ carries coordinates $(p, q)$ and has symplectic measure $dp \wedge dq$.

The UIR of $E(2)$ are $L^2(S^1, d\alpha) \ni \psi(\alpha) \mapsto (U(r, \theta)\psi)(\alpha) = e^{i(r_1 \cos \alpha + r_2 \sin \alpha)}\psi(\alpha - \theta)$. 
Theorem

Given the unit vector $\hat{c} \in \mathbb{R}^2$ and the corresponding subgroup $H_0$, there exists a family of affine sections $\sigma: \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p, q) = (\mathcal{R}(q)(\kappa p + \lambda), q)$, where $\kappa, \lambda \in \mathbb{R}^2$ are constant vectors, and $\hat{c} \cdot \kappa 
eq 0$. 

Theorem

The vectors $\eta_{p, q}$ form a family of coherent states for $E(2)$ which resolves the identity on $L^2(\mathbb{S}^1, d\alpha)$, $I = \int \mathbb{R} \times \mathbb{S}^1 d p d q |\eta_{p, q}\rangle\langle \eta_{p, q}|$, if $\eta(\alpha)$ is admissible in the sense that $\text{supp} \eta \in (\gamma - \pi, \gamma) \mod 2\pi$, and $0 < c_\eta := 2\pi \int_{\mathbb{S}^1} |\eta(q)|^2 \sin(\gamma - q) d q < \infty$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)
Given the unit vector $\hat{c} \in \mathbb{R}^2$ and the corresponding subgroup $H_0$, there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p, q) = (R(q)(\kappa p + \lambda), q)$, where $\kappa, \lambda \in \mathbb{R}^2$ are constant vectors, and $\hat{c} \cdot \kappa \neq 0$.

From the section $\sigma(p, q)$, the representation $U(r, \theta)$, and a vector $\eta \in L^2(\mathbb{S}^1, d\alpha)$, we define the family of states $|\eta_{p, q}\rangle = U(\sigma(p, q))|\eta\rangle$. 
Coherent states for E(2)

Theorem

Given the unit vector $\hat{\mathbf{c}} \in \mathbb{R}^2$ and the corresponding subgroup $H_0$, there exists a family of affine sections $\sigma : \mathbb{R} \times S^1 \to E(2)$ defined as $\sigma(p, q) = (R(q)(\kappa p + \lambda), q)$, where $\kappa, \lambda \in \mathbb{R}^2$ are constant vectors, and $\hat{\mathbf{c}} \cdot \kappa \neq 0$.

From the section $\sigma(p, q)$, the representation $U(r, \theta)$, and a vector $\eta \in L^2(S^1, d\alpha)$, we define the family of states $|\eta_{p, q}\rangle = U(\sigma(p, q))|\eta\rangle$.

Theorem

The vectors $\eta_{p, q}$ form a family of coherent states for E(2) which resolves the identity on $L^2(S^1, d\alpha)$, $I = \int_{\mathbb{R} \times S^1} \frac{dp \, dq}{c_\eta} |\eta_{p, q}\rangle\langle \eta_{p, q}|$, if $\eta(\alpha)$ is admissible in the sense that $\text{supp} \eta \in (\gamma - \pi, \gamma) \mod 2\pi$, and $0 < c_\eta := \frac{2\pi}{\kappa} \int_{S^1} \frac{|\eta(q)|^2}{\sin(\gamma - q)} \, dq < \infty$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil) 
Quantum Localisation on the Circle
Quantisation of classical variables

Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_{f}^{\sigma} = \int_{\mathbb{R} \times \mathbb{S}^1} \frac{dp \, dq}{c_{\eta}} f(p,q) |\eta_{p,q}\rangle \langle \eta_{p,q}|$ to classical observables $f(p,q)$. 

For the Fourier exponential $e_{n}(\alpha) = e^{i n \alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta};\gamma^{*}e_{n})_{\alpha} = 2\pi c_{n}(E_{\eta};\gamma)e^{i n \alpha}$.

For the momentum $f(p,q) = p$, $(A_{p}\psi)(\alpha) = (-i c_{2}(\eta,\gamma)\kappa_{c_{1}(\eta,\gamma)}) \frac{\partial}{\partial \alpha} - \lambda a_{\alpha}) \psi(\alpha)$ for real $\eta$.

For a general polynomial $f(q,p) = \sum_{N}^{k=0} u_{k}(q)p_{k}$ one gets $\sum_{N}^{k=0} a_{k}(\alpha) (-i \frac{\partial}{\partial \alpha})^{k}$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)
Quantisation of classical variables

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R} \times S^1} \frac{dp \, dq}{c_{\eta}} f(p, q) |\eta_{p,q}\rangle \langle \eta_{p,q}|$ to classical observables $f(p, q)$.

- For $f(p, q) = u(q)$ with $u(q + 2\pi) = u(q)$, $A_u$ is the multiplication operator $(A_u \psi)(\alpha) = (E_{\eta;\gamma} \ast u)(\alpha) \psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_{\eta}} \frac{|\eta(\alpha)|^2}{\sin(\gamma - \alpha)}$, $\text{supp} \, E_{\eta;\gamma} \subset (\gamma - \pi, \gamma)$ is a probability distribution on the interval $[-\pi, \pi]$. 
Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A^\sigma_f = \int_{\mathbb{R} \times S^1} \frac{dp\,dq}{c_\eta} f(p,q) |\eta_{p,q}\rangle \langle \eta_{p,q}|$ to classical observables $f(p,q)$.

For $f(p,q) = u(q)$ with $u(q + 2\pi) = u(q)$, $A_u$ is the multiplication operator $(A_u \psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha) \psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta} \frac{|\eta(\alpha)|^2}{\sin(\gamma - \alpha)}$, $\text{supp} \ E_{\eta;\gamma} \subset (\gamma - \pi, \gamma)$ is a probability distribution on the interval $[-\pi, \pi]$.

In particular, for the Fourier exponential $e_n(\alpha) = e^{in\alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)(\alpha) = 2\pi c_n (E_{\eta;\gamma}) e^{in\alpha}$. 
Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A^\sigma_f = \int_{\mathbb{R} \times S^1} \frac{dp \, dq}{c_\eta} f(p, q) |\eta_{p,q}\rangle \langle \eta_{p,q}|$ to classical observables $f(p, q)$.

For $f(p, q) = u(q)$ with $u(q + 2\pi) = u(q)$, $A_u$ is the multiplication operator $(A_u \psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha) \psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta} \frac{|\eta(\alpha)|^2}{\sin(\gamma - \alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma - \pi, \gamma)$ is a probability distribution on the interval $[-\pi, \pi]$.

In particular, for the Fourier exponential $e_n(\alpha) = e^{i n \alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)(\alpha) = 2\pi c_n (E_{\eta;\gamma}) e^{i n \alpha}$.

For the momentum $f(p, q) = p$,

$$(A_p \psi)(\alpha) = \left(-i \frac{c_2(\eta, \gamma)}{\kappa c_1(\eta, \gamma)} \frac{\partial}{\partial \alpha} - \lambda a\right) \psi(\alpha) \text{ for real } \eta.$$
Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R} \times S^1} \frac{dp \, dq}{c_\eta} f(p, q) |\eta_{p,q}\rangle \langle \eta_{p,q}|$ to classical observables $f(p, q)$.

For $f(p, q) = u(q)$ with $u(q + 2\pi) = u(q)$, $A_u$ is the multiplication operator $(A_u \psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha) \psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta} \frac{|\eta(\alpha)|^2}{\sin(\gamma - \alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma - \pi, \gamma)$ is a probability distribution on the interval $[-\pi, \pi]$.

In particular, for the Fourier exponential $e_n(\alpha) = e^{in\alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)(\alpha) = 2\pi c_n (E_{\eta;\gamma}) e^{in\alpha}$.

For the momentum $f(p, q) = p$,$$(A_p \psi)(\alpha) = \left(-i \frac{c_2(\eta, \gamma)}{\kappa c_1(\eta, \gamma)} \frac{\partial}{\partial \alpha} - \lambda a\right) \psi(\alpha) \text{ for real } \eta.$$For a general polynomial $f(q, p) = \sum_{k=0}^{N} u_k(q) p^k$ one gets $\sum_{k=0}^{N} a_k(\alpha)(-i \partial_\alpha)^k$. 
For the $2\pi$-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta, \gamma} * a)(\alpha) = \alpha + 2\pi (1 - \int_{-\pi}^{\alpha} E_{\eta, \gamma}(q) \, dq) - \int_{\gamma-\pi}^{\gamma} q \, E_{\eta, \gamma}(q) \, dq$.
the Angle operator: analytic and numerical results

- For the $2\pi$-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta,\gamma} \ast a)(\alpha) = \alpha + 2\pi (1 - \int_{-\pi}^{\alpha} E_{\eta,\gamma}(q) \, dq) - \int_{\gamma-\pi}^{\gamma} q \, E_{\eta,\gamma}(q) \, dq$.

- We choose a specific section with $\lambda = 0$, $\gamma = \pi/2$ and as fiducial vectors the family $\eta^{(s,\epsilon)}(\alpha)$ of periodic smooth even functions, $\text{supp}\eta = [-\epsilon, \epsilon] \mod 2\pi$, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$
\eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}} \omega_s \left( \frac{\alpha}{\epsilon} \right) \quad \text{where} \quad e_s := \int_{-1}^{1} \, dx \, \omega_s(x).
$$

and

$$
\omega_s(x) = \begin{cases} 
\exp \left( -\frac{s}{1 - x^2} \right) & 0 \leq |x| < 1, \\
0 & |x| \geq 1,
\end{cases}
$$

are smooth and compactly supported test functions.
For the $2\pi$-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta,\gamma} * a)(\alpha) = \alpha + 2\pi(1 - \int_{-\pi}^{\alpha} E_{\eta,\gamma}(q) \, dq) - \int_{\gamma - \pi}^{\gamma} q \, E_{\eta,\gamma}(q) \, dq$.

We choose a specific section with $\lambda = 0$, $\gamma = \pi/2$ and as fiducial vectors the family $\eta^{(s,\epsilon)}(\alpha)$ of periodic smooth even functions, $\text{supp}\eta = [-\epsilon, \epsilon]$ mod $2\pi$, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$\eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}} \omega_s \left( \frac{\alpha}{\epsilon} \right) \quad \text{where} \quad e_s := \int_{-1}^{1} \, dx \, \omega_s(x).$$

and

$$\omega_s(x) = \begin{cases} \exp \left( -\frac{s}{1 - x^2} \right) & 0 \leq |x| < 1, \\ 0 & |x| \geq 1, \end{cases}$$

are smooth and compactly supported test functions.

$$\left(\eta^{(s,\epsilon)}\right)^2(\alpha) \to \delta(\alpha) \quad \text{as} \quad \epsilon \to 0 \quad \text{or} \quad \text{as} \quad s \to \infty.$$
Figure: Plots of $\eta^{(s,\epsilon)}$ for various values of $\tau = \frac{s}{\epsilon^2}$. 

(a) $\tau = 22.22$

(b) $\tau = 81.63$
Figure: Plots of \( E_{\eta(s, \epsilon)}(\alpha) \) for various values of \( \tau = \frac{s}{\epsilon^2} \).

(a) \( \tau = 22.22 \)

(b) \( \tau = 81.63 \)
\( \hat{q} \)

\[ \begin{align*}
\epsilon &= 0.3, \\
\epsilon &= 0.4.
\end{align*} \]

\[ s = 2, \quad s = 3.5555, \quad s = 10, \quad s = 16.5306. \]

\( \tau = 22.22 \) \hspace{1cm} \( \tau = 81.63 \)

**Figure:** Plots of the lower symbol \( \hat{q}(q) \) of the angle operator \( A_A \) for various values of \( \tau = \frac{s}{\epsilon^2} \).
For $\lambda = 0$ and $\psi(\alpha) \in L^2(S^1, d\alpha)$, we find the non-canonical CR $([A_p, A_a]\psi)(\alpha) = -ic (1 - 2\pi E_{\eta;\gamma}(\alpha)) \psi(\alpha)$ where $c := \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)  Quantum Localisation on the Circle
• For $\lambda = 0$ and $\psi(\alpha) \in L^2(S^1, d\alpha)$, we find the non-canonical CR $([A_p, A_a]\psi)(\alpha) = -ic (1 - 2\pi E_{\eta;\gamma}(\alpha)) \psi(\alpha)$ where 

$$c := \frac{c_2(\eta, \gamma)}{\kappa c_1(\eta, \gamma)}$$

• Since $\lim_{\epsilon \to 0} \frac{c_2(\eta^{(s, \epsilon)}, \frac{\pi}{2})}{c_1(\eta^{(s, \epsilon)}, \frac{\pi}{2})} = 1$ and $\lim_{\epsilon \to 0} E_{\eta;\gamma}(\alpha) = \delta(\alpha)$, with the choice $\kappa = 1$ one has, in the limit $\epsilon \to 0$, for $\alpha \in [0, 2\pi)$ mod 2$\pi$, $([A_p, A_a]\psi)(\alpha) = (-i + i2\pi\delta(\alpha)) \psi(\alpha)$. 

For $\lambda = 0$ and $\psi(\alpha) \in L^2(S^1, d\alpha)$, we find the non-canonical CR $([A_p, A_a] \psi)(\alpha) = -ic (1 - 2\pi E_{\eta;\gamma}(\alpha)) \psi(\alpha)$ where 

\[ c := \frac{c_2(\eta, \gamma)}{\kappa c_1(\eta, \gamma)} \]

Since $\lim_{\epsilon \to 0} \frac{c_2(\eta^{(s,\epsilon)}, \frac{\pi}{2})}{c_1(\eta^{(s,\epsilon)}, \frac{\pi}{2})} = 1$ and $\lim_{\epsilon \to 0} E_{\eta,\gamma}(\alpha) = \delta(\alpha)$, with the choice $\kappa = 1$ one has, in the limit $\epsilon \to 0$, for $\alpha \in [0, 2\pi)$ mod $2\pi$, $([A_p, A_a] \psi)(\alpha) = (-i + i2\pi \delta(\alpha)) \psi(\alpha)$.

The uncertainty relation for $A_p$ and $A_a$, with the coherent states $\eta_{p,q}$, is $\Delta A_p \Delta A_a \geq \frac{1}{2} |\langle \eta_{p,q} | [A_p, A_a] | \eta_{p,q} \rangle|$. 
Figure: Plots of the dispersions $\Delta A_a$ and $\Delta A_p$ with respect to the coherent state $|\eta_{p,q}^{(s,\epsilon)}\rangle$ for various values of $\tau = \frac{s}{\epsilon^2}$.
Figure: Plots of the difference L.H.S.-R.H.S. of the uncertainty relation with respect to the coherent state $|\eta_{p,q}^{(s,\epsilon)}\rangle$ for various values of $\tau = \frac{s}{\epsilon^2}$.
Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times S^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $E(2)/H$, where $H$ is a stabilizer subgroup under the coadjoint action of $E(2)$.
Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $E(2)/H$, where $H$ is a stabilizer subgroup under the coadjoint action of $E(2)$.
- The coherent states for $E(2)$ are constructed from a UIR of $E(2) = \mathbb{R}^2 \rtimes SO(2)$ restricted to an affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) \in E(2)$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)  
Quantum Localisation on the Circle
Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder \( \mathbb{R} \times S^1 \) depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset \( E(2)/H \), where \( H \) is a stabilizer subgroup under the coadjoint action of \( E(2) \).
- The coherent states for \( E(2) \) are constructed from a UIR of \( E(2) = \mathbb{R}^2 \rtimes SO(2) \) restricted to an affine section \( \mathbb{R} \times S^1 \ni (p, q) \mapsto \sigma(p, q) \in E(2) \).
- For functions on the cylindric phase space, the corresponding operators and lower symbols are determined. For periodic functions \( f(q) \) of the angular coordinate \( q \), the operators \( A_f \) are multiplication operators whose spectra are given by periodic functions.
We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).

The cylinder $\mathbb{R} \times S^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $E(2)/H$, where $H$ is a stabilizer subgroup under the coadjoint action of $E(2)$.

The coherent states for $E(2)$ are constructed from a UIR of $E(2) = \mathbb{R}^2 \rtimes SO(2)$ restricted to an affine section $\mathbb{R} \times S^1 \ni (p, q) \mapsto \sigma(p, q) \in E(2)$.

For functions on the cylindric phase space, the corresponding operators and lower symbols are determined. For periodic functions $f(q)$ of the angular coordinate $q$, the operators $A_f$ are multiplication operators whose spectra are given by periodic functions.

The angle function $a(\alpha) = \alpha$ is mapped to a SA multiplication angle operator $A_a$ with continuous spectrum.
Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group $E(2)$.
- The cylinder $\mathbb{R} \times S^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $E(2)/H$, where $H$ is a stabilizer subgroup under the coadjoint action of $E(2)$.
- The coherent states for $E(2)$ are constructed from a UIR of $E(2) = \mathbb{R}^2 \rtimes SO(2)$ restricted to an affine section $\mathbb{R} \times S^1 \ni (p, q) \mapsto \sigma(p, q) \in E(2)$.
- For functions on the cylindric phase space, the corresponding operators and lower symbols are determined. For periodic functions $f(q)$ of the angular coordinate $q$, the operators $A_f$ are multiplication operators whose spectra are given by periodic functions.
- The angle function $a(\alpha) = \alpha$ is mapped to a SA multiplication angle operator $A_a$ with continuous spectrum.
- For a particular family of coherent states, it is shown that the spectrum is $[\pi - m(s, \epsilon), \pi + m(s, \epsilon)]$, where $m(s, \epsilon) \to \pi$ as $\epsilon \to 0$ or $s \to \infty$. 

Rodrigo Fresneda (UFABC - São Paulo, Brasil)  
Quantum Localisation on the Circle
Figure: UFABC Campus in Santo André, São Paulo