Quantum Localisation on the Circle

XXth International Conference on Geometry, Integrability and Quantization

Rodrigo Fresneda (UFABC - São Paulo, Brasil)

Varna, 6th of June of 2018

J. Math. Phys., 59(5), 52105 (2018), in collaboration with J.P. Gazeau (Univ. Paris-Diderot) and D. Noguera (CBPF, Rio de Janeiro)

Universidade Federal do ABC

- If $\psi(\alpha)$ is the 2π-periodic wave function on the circle, the quantum angle $\widehat{\alpha}$ cannot be a multiplication operator, $\widehat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.

- If $\psi(\alpha)$ is the 2π-periodic wave function on the circle, the quantum angle $\widehat{\alpha}$ cannot be a multiplication operator, $\widehat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.
- Except if $\widehat{\alpha}$ stands for the 2π-periodic discontinuous angle function,

$$(\widehat{\alpha}\psi)(\alpha) := \left(\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor\right) \psi(\alpha)\,. \tag{1}$$

- If $\psi(\alpha)$ is the 2π-periodic wave function on the circle, the quantum angle $\widehat{\alpha}$ cannot be a multiplication operator, $\widehat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.
- Except if $\widehat{\alpha}$ stands for the 2π-periodic discontinuous angle function,

$$(\widehat{\alpha}\psi)(\alpha) := \left(\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor\right) \psi(\alpha) \,. \tag{1}$$

- However, for $\widehat{\alpha}$ Self-Adjoint (SA), $spec(\widehat{\alpha}) \subset [0, 2\pi]$, the CCR $[\widehat{\alpha}, \widehat{p}_\alpha] = \mathrm{i}\hbar I$ does not hold for SA quantum angular momentum $\widehat{p}_\alpha = -\mathrm{i}\hbar\frac{\partial}{\partial\alpha}$.

- If $\psi(\alpha)$ is the 2π-periodic wave function on the circle, the quantum angle $\widehat{\alpha}$ cannot be a multiplication operator, $\widehat{\alpha}\psi(\alpha) = \alpha\psi(\alpha)$ without breaking periodicity.
- Except if $\widehat{\alpha}$ stands for the 2π-periodic discontinuous angle function,

$$(\widehat{\alpha}\psi)(\alpha) := \left(\alpha - 2\pi \left\lfloor \frac{\alpha}{2\pi} \right\rfloor\right) \psi(\alpha) \,. \tag{1}$$

- However, for $\widehat{\alpha}$ Self-Adjoint (SA), $spec(\widehat{\alpha}) \subset [0, 2\pi]$, the CCR $[\widehat{\alpha}, \widehat{p}_\alpha] = \mathrm{i}\hbar I$ does not hold for SA quantum angular momentum $\widehat{p}_\alpha = -\mathrm{i}\hbar\frac{\partial}{\partial\alpha}$.
- Instead, one has

$$[\widehat{\alpha}, \widehat{p}_\alpha] = i\hbar I \left[1 - 2\pi \sum_n \delta(\alpha - 2n\pi)\right] \,. \tag{2}$$

- This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").

- This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").
- Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.

- This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").
- Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.
- We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).

- This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").
- Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.
- We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).
- Our approach is group theoretical, based on the unitary irreducible representations of the (special) Euclidean group $E(2) = \mathbb{R}^2 \rtimes SO(2)$ (see also S. De Bièvre, Coherent states over symplectic homogeneous spaces).

- This is an old problem (dating back to Dirac "The Quantum Theory of the Emission and Absorption of Radiation").
- Most approaches rely on replacing the angle operator by a quantum version of a smooth periodic function of the classical angle at the cost of losing localisation.
- We revisit the problem of the quantum angle through coherent state (CS) quantisation, which is a particular method belonging to covariant integral quantisation (S.T. Ali, J.-P. Antoine, and J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations*).
- Our approach is group theoretical, based on the unitary irreducible representations of the (special) Euclidean group $E(2) = \mathbb{R}^2 \rtimes SO(2)$ (see also S. De Bièvre, Coherent states over symplectic homogeneous spaces).
- One of our aims is to build acceptable angle operators from the classical angle function through a consistent and manageable quantisation procedure.

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g)\, \mathrm{d}\mu(g)\,, \rho(g) := U(g)\, \rho U^\dagger(g)$$

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g)\,\mathrm{d}\mu(g)\,,\rho(g) := U(g)\,\rho U^\dagger(g)$$

- Then , $R = c_\rho I$, since $U(g_0)\,RU^\dagger(g_0) = \int_G \rho(g_0 g)\,\mathrm{d}\mu(g) = R$

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho\left(g\right) \mathrm{d}\mu\left(g\right), \rho\left(g\right) := U\left(g\right) \rho U^{\dagger}\left(g\right)$$

- Then , $R = c_\rho I$, since $U\left(g_0\right) R U^{\dagger}\left(g_0\right) = \int_G \rho\left(g_0 g\right) \mathrm{d}\mu\left(g\right) = R$
- That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$\int_G \rho\left(g\right) \frac{\mathrm{d}\mu\left(g\right)}{c_\rho} = I\,, \quad c_\rho = \int_G \mathrm{tr}(\rho_0 \rho(g)) \mathrm{d}\mu\left(g\right)$$

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho\left(g\right) \mathrm{d}\mu\left(g\right), \rho\left(g\right) := U\left(g\right) \rho U^{\dagger}\left(g\right)$$

- Then , $R = c_\rho I$, since $U\left(g_0\right) R U^{\dagger}\left(g_0\right) = \int_G \rho\left(g_0 g\right) \mathrm{d}\mu\left(g\right) = R$
- That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$\int_G \rho\left(g\right) \frac{\mathrm{d}\mu\left(g\right)}{c_\rho} = I\,, \quad c_\rho = \int_G \mathrm{tr}(\rho_0 \rho(g)) \mathrm{d}\mu\left(g\right)$$

- This allows an integral quantisation of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g)\, f(g) \frac{\mathrm{d}\mu\left(g\right)}{c_\rho}\,,$$

Covariant integral quantisation - general scheme

UFABC

- Let G be a Lie group with left Haar measure $d\mu(g)$ and $g \mapsto U(g)$ a UIR of G in $\mathcal{H}$. For $\rho \in B(\mathcal{H})$, suppose the following operator is defined in a weak sense:

$$R := \int_G \rho(g)\, \mathrm{d}\mu(g), \rho(g) := U(g)\, \rho U^\dagger(g)$$

- Then , $R = c_\rho I$, since $U(g_0)\, R U^\dagger(g_0) = \int_G \rho(g_0 g)\, \mathrm{d}\mu(g) = R$
- That is, the family of operators $\rho(g)$ provides a resolution of the identity

$$\int_G \rho(g) \frac{\mathrm{d}\mu(g)}{c_\rho} = I\,, \quad c_\rho = \int_G \mathrm{tr}(\rho_0 \rho(g))\mathrm{d}\mu(g)$$

- This allows an integral quantisation of complex-valued functions on the group

$$f \mapsto A_f = \int_G \rho(g)\, f(g) \frac{\mathrm{d}\mu(g)}{c_\rho}\,,$$

- which is covariant in the sense that

$$U(g) A_f U^\dagger(g) = A_{U(g)f}\,,\, (U(g)f)(g') = f(g^{-1}g')$$

- We consider the quantisation of functions on a homogeneous space X, the left coset manifold $X \sim G/H$ for the action of a Lie Group G, where the closed subgroup H is the stabilizer of some point of X.

- We consider the quantisation of functions on a homogeneous space X, the left coset manifold $X \sim G/H$ for the action of a Lie Group G, where the closed subgroup H is the stabilizer of some point of X.
- The interesting case is when X is a symplectic manifold (e.g., co-adjoint orbit of G) and can be viewed as the phase space for the dynamics.

- We consider the quantisation of functions on a homogeneous space X, the left coset manifold $X \sim G/H$ for the action of a Lie Group G, where the closed subgroup H is the stabilizer of some point of X.
- The interesting case is when X is a symplectic manifold (e.g., co-adjoint orbit of G) and can be viewed as the phase space for the dynamics.
- Given a quasi-invariant measure ν on X, one has for a global Borel section $\sigma : X \to G$ a unique quasi-invariant measure $\nu_\sigma(x)$.

- We consider the quantisation of functions on a homogeneous space X, the left coset manifold $X \sim G/H$ for the action of a Lie Group G, where the closed subgroup H is the stabilizer of some point of X.
- The interesting case is when X is a symplectic manifold (e.g., co-adjoint orbit of G) and can be viewed as the phase space for the dynamics.
- Given a quasi-invariant measure ν on X, one has for a global Borel section $\sigma : X \to G$ a unique quasi-invariant measure $\nu_\sigma(x)$.
- Let U be a square-integrable UIR, and ρ_0 a density operator such that $c_\rho := \int_X \mathrm{tr}\,(\rho_0\,\rho_\sigma(x))\,\mathrm{d}\nu_\sigma(x) < \infty$ with $\rho_\sigma(x) := U(\sigma(x))\rho U(\sigma(x))^\dagger$.

- One has the resolution of the identity

$$I = \frac{1}{c_\rho} \int_X \rho_\sigma(x) \, \mathrm{d}\nu_\sigma(x) \, .$$

- One has the resolution of the identity

$$I = \frac{1}{c_\rho} \int_X \rho_\sigma(x) \, \mathrm{d}\nu_\sigma(x) \, .$$

- We then define the quantisation of functions on X as the linear map

$$f \mapsto A^\sigma_f = \frac{1}{c_\rho} \int_X \, f(x) \, \rho_\sigma(x) \, \mathrm{d}\nu_\sigma(x) \, .$$

- One has the resolution of the identity

$$I = \frac{1}{c_\rho} \int_X \rho_\sigma(x)\, \mathrm{d}\nu_\sigma(x)\,.$$

- We then define the quantisation of functions on X as the linear map

$$f \mapsto A_f^\sigma = \frac{1}{c_\rho} \int_X f(x)\, \rho_\sigma(x)\, \mathrm{d}\nu_\sigma(x)\,.$$

- Covariance holds in the sense $U(g) A_f^\sigma U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g) f}$, where $\sigma_g(x) = g\sigma(g^{-1}x)$ with $\mathcal{U}_l(g) f(x) = f\left(g^{-1}x\right)$.
- For $\rho = |\eta\rangle\langle\eta|$, we are working with CS quantisation, where the CS's are defined as $|\eta_x\rangle := |U(\sigma_g(x))\eta\rangle$.

Coherent states for semi-direct product groups

- Let V, $dimV = n$, $S \leq GL(V)$ and $G = V \rtimes S$

- Let V, $dimV = n$, $S \leq GL(V)$ and $G = V \rtimes S$
- Given $k_0 \in V^*$, one can show that

$$H_0 = \{g \in G | (k_0, 0) = \mathrm{Ad}_g^{\#}(k_0, 0)\} = N_0 \rtimes S_0$$

for $(k_0, 0) \in \mathfrak{g}^*$

Coherent states for semi-direct product groups

- Let V, $dimV = n$, $S \leq GL(V)$ and $G = V \rtimes S$
- Given $k_0 \in V^*$, one can show that

$$H_0 = \{g \in G | (k_0, 0) = \mathrm{Ad}^{\#}_{g}(k_0, 0)\} = N_0 \rtimes S_0$$

for $(k_0, 0) \in \mathfrak{g}^*$

- Furthermore, $X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^*\mathcal{O}^*$, $V_0 = T^*_{k_0}\mathcal{O}^*$, is a symplectic manifold with symplectic measure $\mathrm{d}\mu(\mathbf{p}, \mathbf{q})$ which allows the construction of a section $V_0 \times \mathcal{O}^* \ni (\mathbf{p}, \mathbf{q}) \mapsto \sigma(\mathbf{p}, \mathbf{q}) \in G$

UFABC

- Let V, $dimV = n$, $S \leq GL(V)$ and $G = V \rtimes S$
- Given $k_0 \in V^*$, one can show that

$$H_0 = \{g \in G | (k_0, 0) = \mathrm{Ad}^{\#}_g (k_0, 0)\} = N_0 \rtimes S_0$$

  for $(k_0, 0) \in \mathfrak{g}^*$
- Furthermore, $X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^*\mathcal{O}^*$, $V_0 = T^*_{k_0}\mathcal{O}^*$, is a symplectic manifold with symplectic measure $\mathrm{d}\mu(\mathbf{p}, \mathbf{q})$ which allows the construction of a section $V_0 \times \mathcal{O}^* \ni (\mathbf{p}, \mathbf{q}) \mapsto \sigma(\mathbf{p}, \mathbf{q}) \in G$
- Finally, given a UIR χ of V and a UIR L of S, one can construct an irreducible representation $(v, s) \mapsto {}^{\chi L}U(v, s)$ of G induced by the representation $\chi \otimes L$ of $V \rtimes S_0$.

- Let V, $dim V = n$, $S \leq GL(V)$ and $G = V \rtimes S$
- Given $k_0 \in V^*$, one can show that

$$H_0 = \{g \in G | (k_0, 0) = \mathrm{Ad}_g^{\#}(k_0, 0)\} = N_0 \rtimes S_0$$

for $(k_0, 0) \in \mathfrak{g}^*$

- Furthermore, $X = G/H_0 \simeq V_0 \times \mathcal{O}^* \simeq T^*\mathcal{O}^*$, $V_0 = T^*_{k_0}\mathcal{O}^*$, is a symplectic manifold with symplectic measure $\mathrm{d}\mu(\mathbf{p}, \mathbf{q})$ which allows the construction of a section $V_0 \times \mathcal{O}^* \ni (\mathbf{p}, \mathbf{q}) \mapsto \sigma(\mathbf{p}, \mathbf{q}) \in G$
- Finally, given a UIR χ of V and a UIR L of S, one can construct an irreducible representation $(v, s) \mapsto {}^{\chi L}U(v, s)$ of G induced by the representation $\chi \otimes L$ of $V \rtimes S_0$.
- Given $\eta \in \mathcal{H} = L^2(\mathcal{O}^*, d\nu)$, one constructs a family $\eta_{\mathbf{p},\mathbf{q}}$: $\eta_{\mathbf{p},\mathbf{q}}(k) = \left({}^{\chi L}U(\sigma(\mathbf{p}, \mathbf{q}))\eta\right)(k)$

- If one can prove $\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi : \mathcal{O}^* \to \mathbb{C}$ and $0 < c_\eta < \infty$

- If one can prove $\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi:\mathcal{O}^*\to\mathbb{C}$ and $0<c_\eta<\infty$
- we obtain the resolution of the identity $\frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,| = I$

- If one can prove $\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi : \mathcal{O}^* \to \mathbb{C}$ and $0 < c_\eta < \infty$
- we obtain the resolution of the identity $\frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,| = I$
- CS quantisation maps the classical function $f(\boldsymbol{p},\boldsymbol{q}) \in V_0\times\mathcal{O}^*$ to the operator on $\mathcal{H}$ $A_f = \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\,|\eta_{\boldsymbol{p},\boldsymbol{q}}\rangle\,\langle\eta_{\boldsymbol{p},\boldsymbol{q}}|\,f(\boldsymbol{p},\boldsymbol{q})$

- If one can prove $\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi : \mathcal{O}^* \to \mathbb{C}$ and $0 < c_\eta < \infty$
- we obtain the resolution of the identity $\frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,| = I$
- CS quantisation maps the classical function $f(\boldsymbol{p},\boldsymbol{q}) \in V_0\times\mathcal{O}^*$ to the operator on $\mathcal{H}$ $A_f = \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\,|\eta_{\boldsymbol{p},\boldsymbol{q}}\rangle\,\langle\eta_{\boldsymbol{p},\boldsymbol{q}}|\,f(\boldsymbol{p},\boldsymbol{q})$
- The quantisation is covariant ${}^{\chi L}U(g)A_f\,{}^{\chi L}U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g)f}\,,\quad A^{\sigma_g}_f := \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\,\left|\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\right\rangle\left\langle\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\right|f(\boldsymbol{p},\boldsymbol{q})\,,$ with $|\,\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\,\rangle = {}^{\chi L}U(g\sigma(g^{-1}(\boldsymbol{p},\boldsymbol{q})))|\,\eta\,\rangle$

- If one can prove $\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\langle\,\phi\,|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle_{\mathcal{H}}\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,|\,\psi\,\rangle_{\mathcal{H}} = c_\eta\langle\,\phi\,|\,\psi\,\rangle$ where $\phi,\psi:\mathcal{O}^*\to\mathbb{C}$ and $0<c_\eta<\infty$
- we obtain the resolution of the identity $\frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})|\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,\rangle\langle\,\eta_{\boldsymbol{p},\boldsymbol{q}}\,| = I$
- CS quantisation maps the classical function $f(\boldsymbol{p},\boldsymbol{q})\in V_0\times\mathcal{O}^*$ to the operator on $\mathcal{H}$ $A_f = \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\,|\eta_{\boldsymbol{p},\boldsymbol{q}}\rangle\,\langle\eta_{\boldsymbol{p},\boldsymbol{q}}|\,f(\boldsymbol{p},\boldsymbol{q})$
- The quantisation is covariant ${}^{\chi L}U(g)A_f\,{}^{\chi L}U(g)^\dagger = A^{\sigma_g}_{\mathcal{U}_l(g)f}\,,\quad A^{\sigma_g}_f := \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p},\boldsymbol{q})\,\left|\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\right\rangle\,\left\langle\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\right|\,f(\boldsymbol{p},\boldsymbol{q})\,,$ with $|\,\eta^{\sigma_g}_{\boldsymbol{p},\boldsymbol{q}}\,\rangle = {}^{\chi L}U(g\sigma(g^{-1}(\boldsymbol{p},\boldsymbol{q})))|\,\eta\,\rangle$
- The semiclassical portrait of the operator A_f is defined as $\check{f}(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{c_\eta}\int_{V_0\times\mathcal{O}^*} \mathrm{d}\mu(\boldsymbol{p}',\boldsymbol{q}')f(\boldsymbol{p}',\boldsymbol{q}')\left|\left\langle\eta_{\boldsymbol{p}',\boldsymbol{q}'}\middle|\eta_{\boldsymbol{p},\boldsymbol{q}}\right\rangle\right|^2\,.$

The Euclidean group E(2)

- Now $G = \mathrm{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathrm{SO}(2)$, so $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\ \boldsymbol{r} \in \mathbb{R}^2\,,\ \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.

The Euclidean group E(2)

- Now $G = \mathrm{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathrm{SO}(2)$, so $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\ \boldsymbol{r} \in \mathbb{R}^2\,,\ \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.
- $V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{\boldsymbol{k} = \mathcal{R}(\theta)\boldsymbol{k}_0 \in \mathbb{R}^2 \,|\, \mathcal{R}(\theta) \in \mathrm{SO}(2)\} \simeq \mathbb{S}^1$

The Euclidean group E(2)

- Now $G = \mathrm{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathrm{SO}(2)$, so $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\ \boldsymbol{r} \in \mathbb{R}^2\,,\ \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.
- $V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{\boldsymbol{k} = \mathcal{R}(\theta)\boldsymbol{k}_0 \in \mathbb{R}^2 \,|\, \mathcal{R}(\theta) \in \mathrm{SO}(2)\} \simeq \mathbb{S}^1$
- The stabilizer under the coadjoint action $\mathrm{Ad}^{\#}_{\mathrm{E}(2)}$ is $H_0 = \left\{(\boldsymbol{x}, 0) \in \mathrm{E}(2) \,|\, \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0,\ \hat{\boldsymbol{c}} \in \mathbb{R}^2,\ \|\hat{\boldsymbol{c}}\| = 1, \text{ fixed}\right\}$.

The Euclidean group E(2)

- Now $G = \mathsf{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathsf{SO}(2)$, so $\mathsf{E}(2) = \mathbb{R}^2 \rtimes \mathsf{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\ \boldsymbol{r} \in \mathbb{R}^2\,,\ \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.
- $V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{\boldsymbol{k} = \mathcal{R}(\theta)\boldsymbol{k}_0 \in \mathbb{R}^2 \,|\, \mathcal{R}(\theta) \in \mathsf{SO}(2)\} \simeq \mathbb{S}^1$
- The stabilizer under the coadjoint action $\mathrm{Ad}^{\#}_{\mathsf{E}(2)}$ is $H_0 = \left\{(\boldsymbol{x}, 0) \in \mathsf{E}(2) \,|\, \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0,\ \hat{\boldsymbol{c}} \in \mathbb{R}^2,\ \|\hat{\boldsymbol{c}}\| = 1, \text{ fixed}\right\}$.
- The classical phase space $X \equiv T^*\mathbb{S}^1 \simeq (\mathbb{R}^2 \rtimes \mathsf{SO}(2))/H_0 \simeq \mathbb{R} \times \mathbb{S}^1$ carries coordinates (p, q) and has symplectic measure $\mathrm{d}p \wedge \mathrm{d}q$.

- Now $G = \mathrm{E}(2)$, where $V = \mathbb{R}^2$ and $S = \mathrm{SO}(2)$, so $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2) = \{(\boldsymbol{r}, \theta)\,,\, \boldsymbol{r} \in \mathbb{R}^2\,,\, \theta \in [0, 2\pi)\}$, with composition $(\boldsymbol{r}, \theta)(\boldsymbol{r}', \theta') = (\boldsymbol{r} + \mathcal{R}(\theta)\boldsymbol{r}', \theta + \theta')$.
- $V^* = \mathbb{R}^2$, $\mathcal{O}^* = \{\boldsymbol{k} = \mathcal{R}(\theta)\boldsymbol{k}_0 \in \mathbb{R}^2 \,|\, \mathcal{R}(\theta) \in \mathrm{SO}(2)\} \simeq \mathbb{S}^1$
- The stabilizer under the coadjoint action $\mathrm{Ad}^{\#}_{\mathrm{E}(2)}$ is $H_0 = \left\{(\boldsymbol{x}, 0) \in \mathrm{E}(2) \,|\, \hat{\boldsymbol{c}} \cdot \boldsymbol{x} = 0,\ \hat{\boldsymbol{c}} \in \mathbb{R}^2,\ \|\hat{\boldsymbol{c}}\| = 1, \text{ fixed}\right\}$.
- The classical phase space $X \equiv T^*\mathbb{S}^1 \simeq (\mathbb{R}^2 \rtimes \mathrm{SO}(2))/H_0 \simeq \mathbb{R} \times \mathbb{S}^1$ carries coordinates (p, q) and has symplectic measure $\mathrm{d}p \wedge \mathrm{d}q$.
- The UIR of E(2) are $L^2(\mathbb{S}^1, \mathrm{d}\alpha) \ni \psi(\alpha) \mapsto (U(\boldsymbol{r}, \theta)\psi)(\alpha) = e^{\mathrm{i}(r_1 \cos\alpha + r_2 \sin\alpha)}\psi(\alpha - \theta)$.

Theorem

Given the unit vector $\hat{\boldsymbol{c}} \in \mathbb{R}^2$ and the corresponding subgroup H_0, there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p,q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)$, where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\boldsymbol{c}} \cdot \boldsymbol{\kappa} \neq 0$.

Theorem

Given the unit vector $\hat{\mathbf{c}} \in \mathbb{R}^2$ and the corresponding subgroup H_0, there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p, q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)$, where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\mathbf{c}} \cdot \boldsymbol{\kappa} \neq 0$.

From the section $\sigma(p, q)$, the representation $U(\mathbf{r}, \theta)$, and a vector $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we define the family of states
$|\eta_{p,q}\rangle = U(\sigma(p, q))|\eta\rangle$.

Theorem

Given the unit vector $\hat{\mathbf{c}} \in \mathbb{R}^2$ and the corresponding subgroup H_0, there exists a family of affine sections $\sigma : \mathbb{R} \times \mathbb{S}^1 \to E(2)$ defined as $\sigma(p,q) = (\mathcal{R}(q)(\boldsymbol{\kappa} p + \boldsymbol{\lambda}), q)$, where $\boldsymbol{\kappa}, \boldsymbol{\lambda} \in \mathbb{R}^2$ are constant vectors, and $\hat{\mathbf{c}} \cdot \boldsymbol{\kappa} \neq 0$.

From the section $\sigma(p,q)$, the representation $U(\mathbf{r},\theta)$, and a vector $\eta \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we define the family of states $|\eta_{p,q}\rangle = U(\sigma(p,q))|\eta\rangle$.

Theorem

The vectors $\eta_{p,q}$ form a family of coherent states for $E(2)$ which resolves the identity on $L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, $I = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} \,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$, if $\eta(\alpha)$ is <u>admissible</u> in the sense that $\operatorname{supp}\eta \in (\gamma-\pi,\gamma) \bmod 2\pi$, and $0 < c_\eta := \frac{2\pi}{\kappa} \int_{\mathbb{S}^1} \frac{|\eta(q)|^2}{\sin(\gamma - q)}\,\mathrm{d}q < \infty$.

Quantisation of classical variables

UFABC

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.
- For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)\,(\alpha)\,\psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\,\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma-\pi,\gamma)$ is a probability distribution on the interval $[-\pi,\pi]$.

Quantisation of classical variables

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \dfrac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.
- For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)\,(\alpha)\,\psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\,\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma-\pi,\gamma)$ is a probability distribution on the interval $[-\pi,\pi]$.
- In particular, for the Fourier exponential $e_n(\alpha) = e^{\mathrm{i}n\alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)\,(\alpha) = 2\pi c_n\,(E_{\eta;\gamma})\;e^{\mathrm{i}n\alpha}$.

Quantisation of classical variables

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A^\sigma_f = \int_{\mathbb{R}\times\mathbb{S}^1} \dfrac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.
- For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)\,(\alpha)\,\psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta}\,\frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma-\pi,\gamma)$ is a probability distribution on the interval $[-\pi,\pi]$.
- In particular, for the Fourier exponential $e_n(\alpha) = e^{\mathrm{i}n\alpha}$, $n\in\mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)\,(\alpha) = 2\pi c_n\,(E_{\eta;\gamma})\;e^{\mathrm{i}n\alpha}$.
- For the momentum $f(p,q) = p$,
$$(A_p\psi)\,(\alpha) = \left(-\mathrm{i}\frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}\frac{\partial}{\partial\alpha} - \lambda a\right)\psi\,(\alpha) \text{ for real } \eta.$$

Quantisation of classical variables

UFABC

- Given the family of coherent states $|\eta_{p,q}\rangle$, we apply the linear map $f \mapsto A_f^\sigma = \int_{\mathbb{R}\times\mathbb{S}^1} \frac{\mathrm{d}p\,\mathrm{d}q}{c_\eta} f(p,q)\,|\,\eta_{p,q}\,\rangle\langle\,\eta_{p,q}\,|$ to classical observables $f(p,q)$.
- For $f(p,q) = u(q)$ with $u(q+2\pi) = u(q)$, A_u is the multiplication operator $(A_u\psi)(\alpha) = (E_{\eta;\gamma} * u)(\alpha)\,\psi(\alpha)$ where $E_{\eta;\gamma}(\alpha) := \frac{2\pi}{\kappa c_\eta} \frac{|\eta(\alpha)|^2}{\sin(\gamma-\alpha)}$, supp $E_{\eta;\gamma} \subset (\gamma-\pi,\gamma)$ is a probability distribution on the interval $[-\pi,\pi]$.
- In particular, for the Fourier exponential $e_n(\alpha) = e^{\mathrm{i}n\alpha}$, $n \in \mathbb{Z}$, the above expression is $(E_{\eta;\gamma} * e_n)(\alpha) = 2\pi c_n (E_{\eta;\gamma})\ e^{\mathrm{i}n\alpha}$.
- For the momentum $f(p,q) = p$,
$$(A_p\psi)(\alpha) = \left(-\mathrm{i}\frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}\frac{\partial}{\partial\alpha} - \lambda a\right)\psi(\alpha) \text{ for real } \eta.$$
- For a general polynomial $f(q,p) = \sum_{k=0}^N u_k(q)\,p^k$ one gets $\sum_{k=0}^N a_k(\alpha)(-\mathrm{i}\partial_\alpha)^k$.

- For the 2π-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta,\gamma} * a)(\alpha) = \alpha + 2\pi(1 - \int_{-\pi}^{\alpha} E_{\eta;\gamma}(q)\,\mathrm{d}q) - \int_{\gamma-\pi}^{\gamma} q\, E_{\eta,\gamma}(q)\,\mathrm{d}q\,.$

- For the 2π-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta,\gamma} * a)(\alpha) = \alpha + 2\pi(1 - \int_{-\pi}^{\alpha} E_{\eta;\gamma}(q)\,\mathrm{d}q) - \int_{\gamma-\pi}^{\gamma} q\, E_{\eta,\gamma}(q)\,\mathrm{d}q\,.$
- We choose a specific section with $\lambda = 0$, $\gamma = \pi/2$ and as fiducial vectors the family $\eta^{(s,\epsilon)}(\alpha)$ of periodic smooth even functions, $\mathrm{supp}\eta = [-\epsilon, \epsilon]$ mod 2π, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

$$\eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}}\,\omega_s\left(\frac{\alpha}{\epsilon}\right) \quad \text{where} \quad e_s := \int_{-1}^{1} \mathrm{d}x\, \omega_s(x)\,.$$

and

$$\omega_s(x) = \begin{cases} \exp\left(-\dfrac{s}{1-x^2}\right) & 0 \leq |x| < 1\,, \\ 0 & |x| \geq 1\,, \end{cases}$$

are smooth and compactly supported test functions.

- For the 2π-periodic and discontinuous angle function $a(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, we get the multiplication operator $(E_{\eta,\gamma} * a)(\alpha) = \alpha + 2\pi(1 - \int_{-\pi}^{\alpha} E_{\eta;\gamma}(q)\,\mathrm{d}q) - \int_{\gamma-\pi}^{\gamma} q\, E_{\eta,\gamma}(q)\,\mathrm{d}q\,.$
- We choose a specific section with $\lambda = 0$, $\gamma = \pi/2$ and as fiducial vectors the family $\eta^{(s,\epsilon)}(\alpha)$ of periodic smooth even functions, $\mathrm{supp}\eta = [-\epsilon, \epsilon] \bmod 2\pi$, parametrized by $s > 0$ and $0 < \epsilon < \pi/2$,

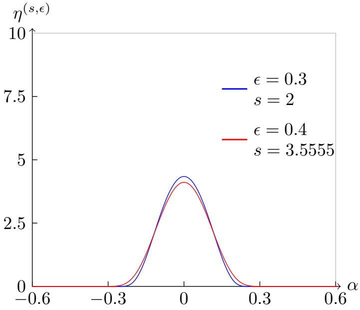
$$\eta^{(s,\epsilon)}(\alpha) = \frac{1}{\sqrt{\epsilon e_{2s}}}\,\omega_s\left(\frac{\alpha}{\epsilon}\right) \quad \text{where} \quad e_s := \int_{-1}^{1} \mathrm{d}x\,\omega_s(x)\,.$$

and

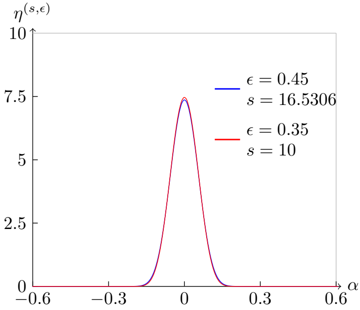
$$\omega_s(x) = \begin{cases} \exp\left(-\dfrac{s}{1-x^2}\right) & 0 \leq |x| < 1\,, \\ 0 & |x| \geq 1\,, \end{cases}$$

are smooth and compactly supported test functions.

- $\left(\eta^{(s,\epsilon)}\right)^2(\alpha) \to \delta(\alpha)$ as $\epsilon \to 0$ or as $s \to \infty$.

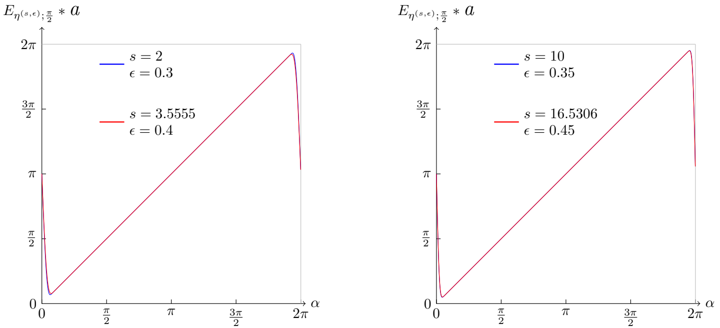


(a) $\tau = 22.22$



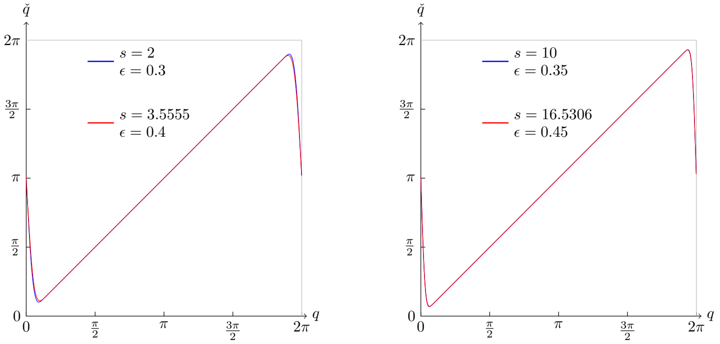
(b) $\tau = 81.63$

Figure: Plots of $\eta^{(s,\epsilon)}$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.



(a) $\tau = 22.22$ (b) $\tau = 81.63$

Figure: Plots of $\left(E_{\eta^{(s,\epsilon)};\frac{\pi}{2}} * a\right)(\alpha)$ for various values of $\tau = \frac{s}{\epsilon^2}$.



(a) $\tau = 22.22$ (b) $\tau = 81.63$

Figure: Plots of the lower symbol $\check{q}(q)$ of the angle operator A_a for various values of $\tau = \dfrac{s}{\epsilon^2}$.

Angle-angular momentum: commutation relations and Heisenberg inequality

UFABC

- For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we find the non-canonical CR $\left([A_p, A_{\mathsf{a}}]\psi\right)(\alpha) = -\mathrm{i}\mathsf{c}\left(1 - 2\pi E_{\eta;\gamma}(\alpha)\right)\psi(\alpha)$ where $\mathsf{c} := \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}$

Angle-angular momentum: commutation relations and Heisenberg inequality

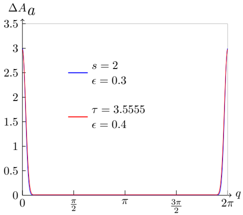
UFABC

- For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we find the non-canonical CR $([A_p, A_{\mathfrak{a}}]\psi)(\alpha) = -\mathrm{i}\mathrm{c}\,(1 - 2\pi E_{\eta;\gamma}(\alpha))\,\psi(\alpha)$ where $\mathrm{c} := \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}$
- Since $\lim_{\epsilon\to 0} \frac{c_2(\eta^{(s,\epsilon)},\frac{\pi}{2})}{c_1(\eta^{(s,\epsilon)},\frac{\pi}{2})} = 1$ and $\lim_{\epsilon\to 0} E_{\eta;\gamma}(\alpha) = \delta(\alpha)$, with the choice $\kappa = 1$ one has, in the limit $\epsilon \to 0$, for $\alpha \in [0, 2\pi)$ mod 2π, $([A_p, A_{\mathfrak{a}}]\psi)(\alpha) = (-\mathrm{i} + \mathrm{i}2\pi\delta(\alpha))\,\psi(\alpha)$.

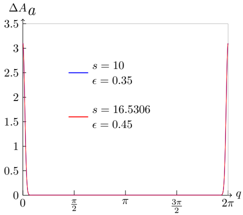
Angle-angular momentum: commutation relations and Heisenberg inequality

UFABC

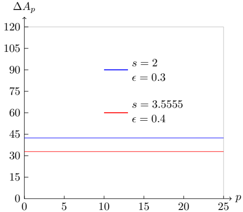
- For $\lambda = 0$ and $\psi(\alpha) \in L^2(\mathbb{S}^1, \mathrm{d}\alpha)$, we find the non-canonical CR $\left([A_p, A_{\mathsf{a}}]\psi\right)(\alpha) = -\mathrm{i}\mathrm{c}\left(1 - 2\pi E_{\eta;\gamma}(\alpha)\right)\psi(\alpha)$ where $\mathrm{c} := \frac{c_2(\eta,\gamma)}{\kappa c_1(\eta,\gamma)}$
- Since $\lim_{\epsilon\to 0} \frac{c_2(\eta^{(s,\epsilon)}, \frac{\pi}{2})}{c_1(\eta^{(s,\epsilon)}, \frac{\pi}{2})} = 1$ and $\lim_{\epsilon \to 0} E_{\eta;\gamma}(\alpha) = \delta(\alpha)$, with the choice $\kappa = 1$ one has, in the limit $\epsilon \to 0$, for $\alpha \in [0, 2\pi)$ mod 2π, $\left([A_p, A_{\mathsf{a}}]\psi\right)(\alpha) = \left(-\mathrm{i} + \mathrm{i}2\pi\delta(\alpha)\right)\psi(\alpha)$.
- The uncertainty relation for A_p and A_{a}, with the coherent states $\eta_{p,q}$, is $\Delta A_p\, \Delta A_{\mathsf{a}} \geqslant \frac{1}{2}\left|\langle\, \eta_{p,q} \,|[A_p, A_{\mathsf{a}}]|\, \eta_{p,q}\,\rangle\right|$.



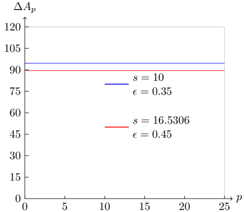
(a) $\tau = 22.22$



(b) $\tau = 81.63$

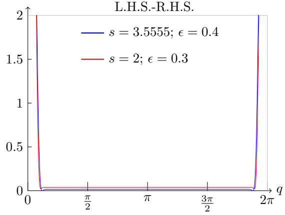


(c) $\tau = 22.22$

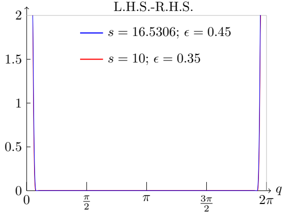


(d) $\tau = 81.63$

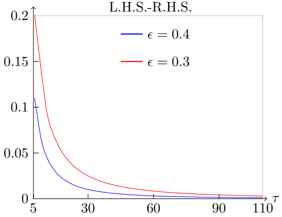
Figure: Plots of the dispersions ΔA_a and ΔA_p with respect to the coherent state $|\eta_{p,q}^{(s,\epsilon)}\rangle$ for various values of $\tau = \dfrac{s}{\epsilon^2}$.



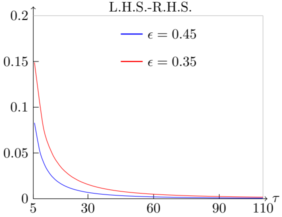
(a) $\tau = 22.22$



(b) $\tau = 81.63$



(c) $q = 1$



(d) $q = 1$

Figure: Plots of the difference L.H.S.-R.H.S. of the uncertainty relation with respect to the coherent state $|\eta_{p,q}^{(s,\epsilon)}\rangle$ for various values of $\tau = \frac{s}{\epsilon^2}$.

Conclusions

UFABC

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).

Conclusions

UFABC

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $\mathrm{E}(2)/H$, where H is a stabilizer subgroup under the coadjoint action of E(2).

Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $\mathrm{E}(2)/H$, where H is a stabilizer subgroup under the coadjoint action of E(2).
- The coherent states for E(2) are constructed from a UIR of $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ restricted to an affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) \in \mathrm{E}(2)$.

Conclusions

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $\mathsf{E}(2)/H$, where H is a stabilizer subgroup under the coadjoint action of E(2).
- The coherent states for E(2) are constructed from a UIR of $\mathsf{E}(2) = \mathbb{R}^2 \rtimes \mathsf{SO}(2)$ restricted to an affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) \in \mathsf{E}(2)$.
- For functions on the cylindric phase space, the corresponding operators and lower symbols are determined . For periodic functions $f(q)$ of the angular coordinate q, the operators A_f are multiplication operators whose spectra are given by periodic functions.

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $\mathrm{E}(2)/H$, where H is a stabilizer subgroup under the coadjoint action of E(2).
- The coherent states for E(2) are constructed from a UIR of $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ restricted to an affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) \in \mathrm{E}(2)$.
- For functions on the cylindric phase space, the corresponding operators and lower symbols are determined . For periodic functions $f(q)$ of the angular coordinate q, the operators A_f are multiplication operators whose spectra are given by periodic functions.
- The angle function $\mathsf{a}(\alpha) = \alpha$ is mapped to a SA multiplication angle operator A_{a} with continuous spectrum.

Conclusions

UFABC

- We have presented a picture of a quantisation based on the resolution of the identity provided by coherent states for the special Euclidean group E(2).
- The cylinder $\mathbb{R} \times \mathbb{S}^1$ depicts the classical phase space of the motion of a particle on a circle, and is mathematically realized as the left coset $\mathrm{E}(2)/H$, where H is a stabilizer subgroup under the coadjoint action of E(2).
- The coherent states for E(2) are constructed from a UIR of $\mathrm{E}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$ restricted to an affine section $\mathbb{R} \times \mathbb{S}^1 \ni (p, q) \mapsto \sigma(p, q) \in \mathrm{E}(2)$.
- For functions on the cylindric phase space, the corresponding operators and lower symbols are determined . For periodic functions $f(q)$ of the angular coordinate q, the operators A_f are multiplication operators whose spectra are given by periodic functions.
- The angle function $a(\alpha) = \alpha$ is mapped to a SA multiplication angle operator A_a with continuous spectrum.
- For a particular family of coherent states, it is shown that the spectrum is $[\pi - m(s, \epsilon), \pi + m(s, \epsilon)]$, where $m(s, \epsilon) \to \pi$ as $\epsilon \to 0$ or $s \to \infty$.

obrigado!

UFABC

Figure: UFABC Campus in Santo André, São Paulo