Complete integrability and separability in Kerr-NUT-AdS spacetimes

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“Black Holes, Hidden Symmetry and Complete Integrability”

V.F., Pavel Krtous and David Kubiznak

Prepared for Living Reviews in Relativity, arXiv:1705.05482 (2017)

“Separation of variables in Maxwell equations in Plebanski-Demianski spacetime”, V. F, P. Krtouš and D. Kubizňák, 2018. 7 pp. e-Print: **arXiv:1802.09491;** accepted by PRD RC.

“Separation of Maxwell equations in Kerr-NUT-(A)dS spacetimes”, P. Krtouš, V.F. And D. Kubizňák, 2018. 33 pp. , e-Print: **arXiv:1803.02485**

“Massive Vector Fields in Kerr-NUT-(A)dS Spacetimes: Separability and Quasinormal Modes”, V.P., P. Krtouš, D. Kubizňák, and J. E. Santo, 2018. 5 pp. e-Print: **arXiv:1804.00030;** (accepted to PRL).

Main result: Properties of higher dimensional (Kerr-NUT-(A)dS) BHs and their 4D ‘cousins’ are very similar. This is also true for their off-shell extensions.

Geodesic equations in these metrics give a wide (infinite) class of new examples of completely integrable dynamical systems.

$$D = 2n + \varepsilon$$

Generator of Symmetries

Principal Closed Conformal KY Tensor

2-form $h_{[ab]}$ with the following properties:

(i) Non-degenerate (maximal matrix rank, $2n$);

(ii) Closed $dh = 0, \quad h = db$;

(iii) Conformal KY tensor: $\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a$,

$\xi_a = \frac{1}{D-1}\nabla^b h_{ba}$, ξ_a is a primary Killing vector

For briefness, we call this object a

"PRINCIPAL TENSOR"

All Kerr-NUT-AdS metrics and their canonical extensions in any number of ST dimensions possess a PRINCIPAL TENSOR

(V.F.&Kubiznak '07)

Uniqueness Theorem

A solution of Einstein equations with the cosmological constant, which possesses a PRINCIPAL TENSOR is a Kerr-NUT-AdS metric

(Houri,Oota&Yasui ’07 ’09; Krtous, V.F. .&Kubiznak ’08;)

Geodesic equations in ST with a PRINCIPAL TENSOR are completely integrable in Liouville sense. Important field equations allow a complete separation of variables.

In the rest of the talk I shall explain why it happens.

Liouville theorem: Dynamical equations in 2N dimensional phase space are completely integrable if there exist N independent commuting integrals of motion.

This means that a solution of equations of motion of such a system can be obtained by quadratures, i.e. by a finite number of algebraic operations and integrations.

Relativistic Particle as a Dynamical System

Preferable coordinates in the phase space:

$(x^a, p_a = g_{ab}\dot{x}^b), \;\; H(p,x) = \frac{1}{2} g^{ab} p_a p_b .$

Monomial in momenta integrals of motion

$\mathcal{K} = K^{a\ldots b} p_a \ldots p_b$ imply that $K_{a\ldots b}$ is a Killing tensor:

$$K_{(a\ldots b;c)} = 0$$

$\{\mathcal{K}_1, \mathcal{K}_2\} = 0 \Leftrightarrow \;\; [K_1, K_2] = 0.$

Motion of particle in D-dimensional ST is completely integrable if there exist D independent commuting Killing tensors (and vectors)

Metric is a best known (trivial) example of rank 2 Killing tensor

Killing vectors are directly connected with ST symmetries. They generate explicit symmetry transformations. The corresponding conserved quantities are linear in momentum. Symmetries of ST, which are responsible for integrals of moton higher order in momentum, are called hidden.

Higher dimensions

Denote $D = 2n + \varepsilon$.

D-dimensional Kerr-NUT-(A)dS metric has $n + \varepsilon$ Killing vectors. For complete integrability of geodesic equations one needs n more integrals of motion. One is trivial, the metric, so that one needs $n - 1$ new non-trivial integrals of motion.

SYMMETRY GENERATION: GENERAL SCHEME

Killing-Yano family

Let ω be p-form on the Riemannian manifold.

$$\nabla_X \omega = \frac{1}{p+1} X \bullet (\nabla \wedge \omega) + \frac{1}{D-p+1} X \wedge (\nabla \bullet \omega) \; [+(\ldots)]$$

If $(\ldots) = 0$ ω is a conformal KY tensor.

$$\nabla_X (*\omega) = \frac{1}{p_* + 1} X \bullet (\nabla \wedge *\omega) + \frac{1}{D - p_* + 1} X \wedge (\nabla \bullet *\omega),$$

$p_* = D - p \Rightarrow *\omega$ is also a conformal KY tensor;

If $\delta\omega \equiv -\nabla \bullet \omega = (\ldots) = 0$ ω is a KY tensor;

If $\mathrm{d}\omega \equiv \nabla \wedge \omega = (\ldots) = 0$ ω is a closed conformal KY tensor; $\omega = d\beta$.

Special case: closed conformal rank 2

KY tensor (Principal tensor) h

$$\nabla_X h = \frac{1}{D-1} X \wedge (\nabla \cdot h)$$

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \quad \xi_a = \tfrac{1}{D-1} \nabla^b h_{ba}$$

Properties of CKY tensor

Hodge dual of CKY tensor is CKY tensor

Hodge dual of CCKY tensor is KY tensor;

Hodge dual of KY tensor is CCKY tensor;

External product of two CCKY tensors is a CCKY tensor

(Krtous,Kubiznak,Page &V.F. '07; V.F. '07)

Let $f^{(1)}$ and $f^{(2)}$ are two KY tensors of rank s. Then $k^{ab} = f^{(1)ac_1 \dots c_{s-1}} f^{(2)b}{}_{c_1 \dots c_{s-1}}$ is a rank-2 Killing tensor.

Killing-Yano Tower

Killing-Yano Tower

$CCKY: \; h \;\Rightarrow\; h^{\wedge j} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j\ times}$

KY tensors: $k_j = {}^{*}h^{\wedge j}$

Killing tensors: $K_j = k_j \bullet k_j$

Primary Killing vector: $\xi_{(0)a} = \frac{1}{D-1} \nabla^b h_{ba}$

Secondary Killing vectors: $\xi_{(j)} = K_j \bullet \xi_{(0)}$

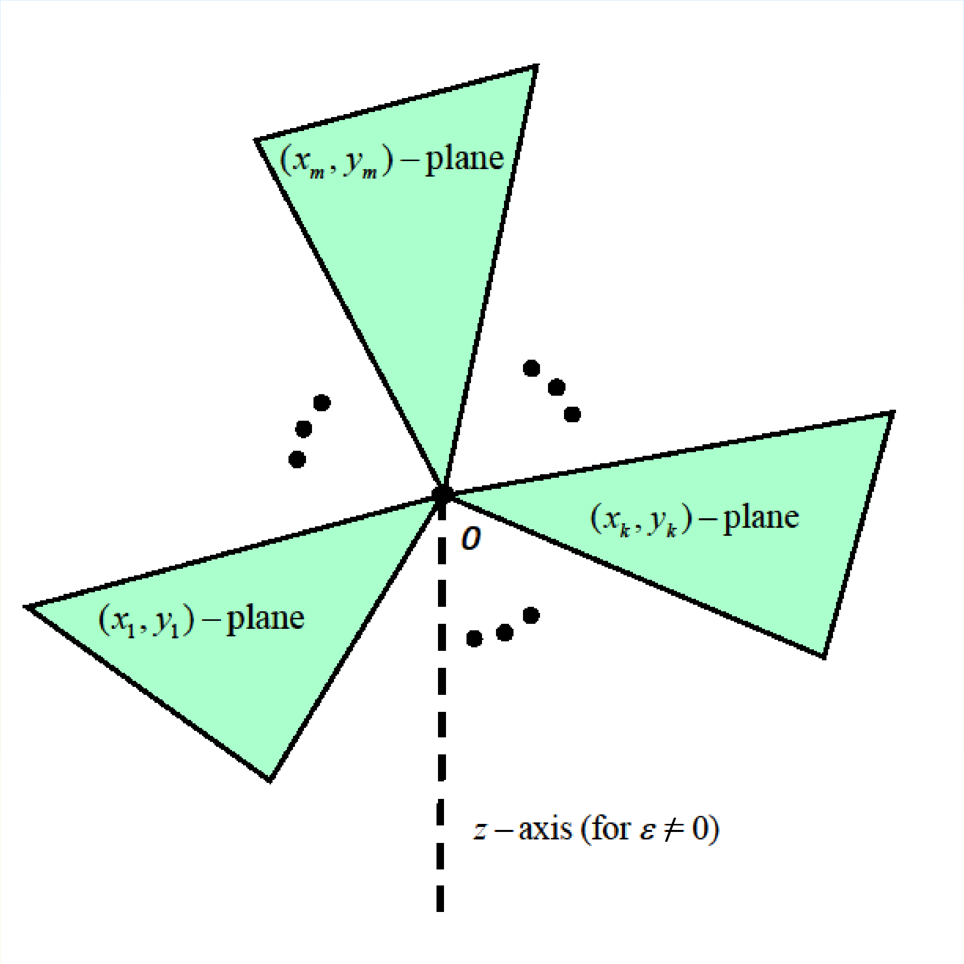
Total number of conserved quantities:

$$(n+\varepsilon)+(n-1)+1=2n+\varepsilon=D$$

KV *KT* *g*

The integrals of motion are functionally independent and in involution. In the presence of the principal tensor the geodesic equations ST are completely integrable.

Canonical Coordinates



$$Q_{ab} = h_a{}^c h_{bc}\ ,\ \ Q \cdot e_\mu = x_\mu^2 e_\mu ,$$

$$h \cdot e_\mu = x_\mu \hat{e}_\mu ,\quad Q \cdot \hat{e}_\mu = x_\mu^2 \hat{e}_\mu ,$$

Darboux frame: $D = 2n + \varepsilon$,

$h = \Sigma_{\mu=1}^{n} x_\mu e_\mu \wedge \hat{e}_\mu$,

$g = \Sigma_{\mu=1}^{n} (e_\mu e_\mu + \hat{e}_\mu \hat{e}_\mu) + \varepsilon \hat{e}_0 \hat{e}_0$

Principal tensor *h* is non-denerate: (i) It has *n* different 2-eigenplanes; (ii) x_μ are different and functionally independent in a domain *U* near given point *p*; $\Rightarrow$ (iii) x_μ can be used as *n* coordinates in this domain.

$(n+\varepsilon)$ prime and secondary Killing vectors $\xi_{(j)}$ are commuting. Moreover one has $L_{\xi_{(j)}} h = 0 \;\Rightarrow\; \xi^{a}_{(j)} x_{\mu} = 0.$

According to Frobeneus theorem, there exist local foliation such that $l_{(i)}$ are tangent to *n*-dimensional surfaces $x_{\mu} = const,$ and one can introduce coordinates ψ_{i} such that $l^{a}_{(i)} \partial_{a} = \partial_{\psi_{\mathrm{i}}}.$

Thus non-degenerate principal tensor determines n essential coordinates x_μ and $n+\varepsilon$ Killing coordinates ψ_i. They together provide one with a set of D canonical coordinates.

$$g = \sum_\mu (e^\mu e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon e^{n+1} e^{n+1},$$

$$h = \sum_\mu x^\mu e^\mu \wedge e^{\hat{\mu}}$$

The components of Killing tower objects in such a basis are polynomials in x^μ.

"Off-shell" metrics, which admit the Principal Tensor, allow complete description

$$ds^2 = \sum_{\mu=1}^{n} [\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} (\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j)^2]$$

$$+ \varepsilon \frac{c}{A^{(n)}} (\sum_{k=0}^{n} A^{(k)} d\psi_k)^2 ,$$

$$U_\mu = \prod_{\nu \neq \mu} (x_\nu^2 - x_\mu^2), \quad X_\mu = X_\mu(x_\mu),$$

$$\prod_{\nu=1}^{n} (1 + t x_\nu^2) = \sum_{j=0}^{n} t^j A^{(j)} ,$$

$$(1 + t x_\mu^2)^{-1} \prod_{\nu=1}^{n} (1 + t x_\nu^2) = \sum_{k=0}^{n-1} t^k A_\mu^{(k)} .$$

Houri, Oota, and Yasui [PLB (2007); JP A41 (2008)] proved this result under additional assumptions: $L_\xi g = 0$ and $L_\xi h = 0$. Later Krtous, V.F., Kubiznak [arXiv:0804.4705 (2008)] and Houri, Oota, and Yasui [arXiv:0805.3877 (2008)] proved this without additional assumptions.

Principal tensor: $h = db$,

$$b = \frac{1}{2}\sum_{k=0}^{n-1} A^{(k+1)}(x_\mu)\, d\psi_k .$$

Solutions of the Einstein equations with the cosmological constant ("On-shell" metrics)

For D even: $X_\mu = -2b_\mu x_\mu + \lambda J(x_\mu^2)$,

$$J(x^2) = \prod_{\nu=1}^{n} (a_\nu^2 - x^2);$$

A similar expression for D odd.

This is nothing but the Kerr-NUT-(A)dS metric, written in special (canonical) coordinates.

These metrics are of type D in the higher dimensional algebraic classification, developed by Coley, Pravda and Pravdova.

The geodesic equations in the off-shell Kerr-NUT-(A)dS metrics are completely integrable.

Hamilton-Jacobi, Klein-Gordon and massive Dirac equations allow complete separation of variables.

Separability of the Hamilton–Jacobi equation in canonical coordinates

$$\frac{\partial S}{\partial \lambda} + g^{ab}\, \partial_a S\, \partial_b S = 0$$

$$S = -m^2\lambda/2 + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{n-1+\varepsilon} L_k \psi_k;$$

$(S_\mu{}')^2 = \dfrac{Y_\mu}{X_\mu^2}$, Y_μ is a known function of x_μ and conserved quantities.

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007)

Separability of the Klein–Gordon equation

$$(\Box - m^2)\Phi = 0$$

,

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007)

$$K_0 = \nabla_a (g^{ab} \nabla_b); \quad K_j = -\nabla_a (k_j^{ab} \nabla_b); \quad L_j = -i\, \xi_j^a \nabla_a \,.$$

$$[K_j, K_k] = [K_j, L_k] = [L_j L_k] = 0;$$

[Sergeev and Krtous (2008); Kolar and Krtous (2015)]

$$K_j \Phi = \kappa_j \Phi, \quad \xi_j \Phi = \lambda_j \Phi;$$

$$\Phi = \prod_{\mu} R_{\mu} \prod_{k=0}^{n-1+\varepsilon} \exp(i\lambda_k \psi_k);$$

$$(X_{\mu} R_{\mu}')' + \varepsilon \frac{X_{\mu}}{x_{\mu}} R_{\mu}' + \frac{Y_{\mu}}{X_{\mu}^2} R_{\mu} = 0.$$

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007)

FURTHER DEVELOPMENTS

- Parallel transport of frames along geodesics;
- Separation of variables in Dirac equations;
- Lax-pairs for integrable geodesics in Kerr-NUT-(A)dS;
- Generalized Kerr-NUT-(A)dS metrics: Cases when the principal tensor becomes degenerate;
- Deformed and twisted black holes with NUTs;
- Separation of special type gravitational perturbation equations;
- Weakly charged (generalized) Kerr-NUT-(A)dS STs;
- Generalization to the geometries with torsion;
- Non-empty BH solutions.

RECENT DEVELOPMENTS

Separability of HD Maxwell equations

Maxwell equations in 4D type D ST (i) can be decoupled, and (ii) the decoupled equations allow complete separation of variables.

[S.A. Teukolsky, Phys.Rev.Lett., 29, 1114-1118 (1972)]

Method: Newman-Penrose equations;

Special choice of null frames;

$\mathbf{F} = F \pm i * F \quad (anti) - self\text{-}dual$ field

Remarkable progress by Oleg Lunin:

"*Maxwell's equations in the Myers-Perry geometry*", *JHEP* 1712 (2017) 138.

He proposed a special ansatz for em potential in canonical coordinates and demonstrated that Maxwell equations in Kerr-deSitter spacetime are separable.

Our recent results in:

V.F, P.Krtouš andD.Kubizňák,e-Print:arXiv:1802.09491;
accepted by PRD RC.

P.Krtouš, V.F. and D.Kubizňák, e-Print:arXiv:1803.02485.

(i) Covariant description;

(ii) Ansatz for the field in terms of principal tensor **only**;

(iii) Analitical proof of separability;

(iv) Results are valid for any metric which admits the principal tensor $\Rightarrow$ For off-shell Kerr-NUT-(A)dS STs.

Polarization tensor: $\mathbf{B} = \dfrac{1}{1 + i\mu\,\mathbf{h}}$,

$$(g_{ab} + i\mu h_{ab})B^{bc} = \delta_a^c,$$

$$\mathbf{A} = \mathbf{B}\,\nabla Z, \quad Z = \prod_{\mu} R_{\mu} \prod_{k=0}^{n-1+\varepsilon} \exp(i\lambda_k \psi_k).$$

(i) Lorenz condition: $\nabla_a A^a = 0 \Rightarrow$ second order partial differential equation for Z, $DZ = 0$, which allows the separation of variables;

(ii) Maxwell equations $\nabla_b F^{ab} = 0$ in the Lorenz gauge can be written in the form $B^{ab}\nabla_b \tilde{D}Z = 0$. This equation is separable. The separated equations are the same as for the Lorenz equation, with an additional condition that one of sepation constantsis put to zero.

Massive vector field in Kerr-NUT-(A)dS spacetime

Proca (1936) equation: $\nabla_b F^{ab} + m^2 A^a = 0$

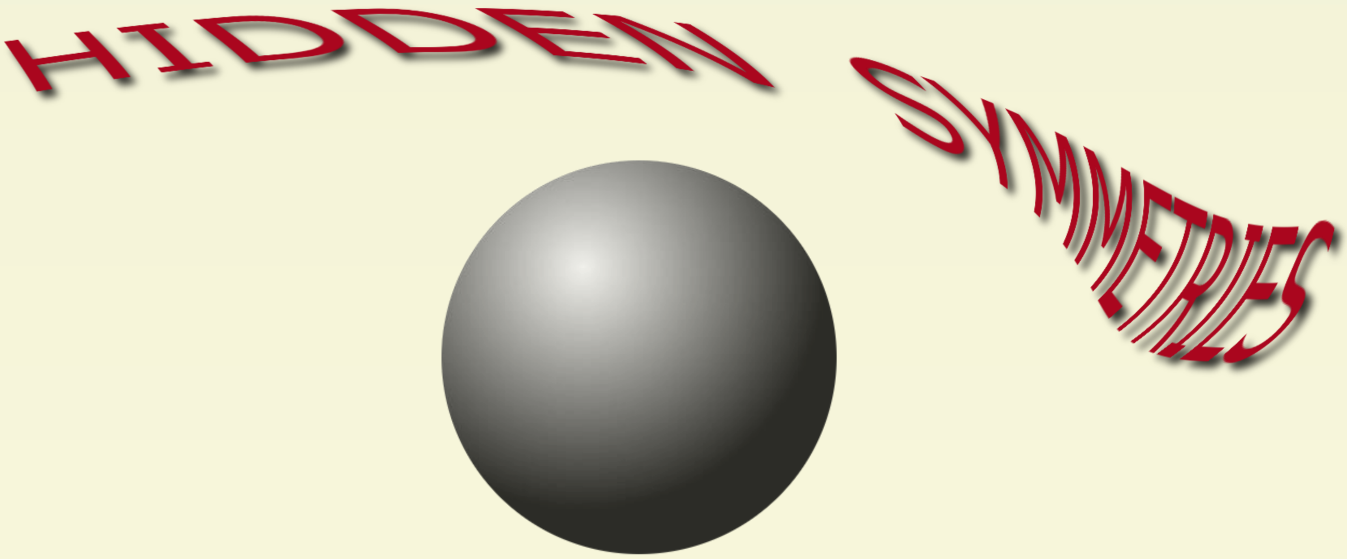
implies Lorenz equation $\nabla_a A^a = 0$

"Constraining the mass of dark photons and axion-like particles through black-hole superradiance",V. Cardoso Ó. Dias,G. Hartnett,M. Middleton,P. Pani,and J.Santos, JCAP 1803 (2018) 043,1475-7516,arXiv:1801.01420

"The extension of the above results to massive vectors is nontrivial. In particular, the corresponding linearized evolution equation do not seem to separate."

In our recent paper we prove separability of Proca equations in off-shell Kerr-NUT-(A)dS spacetime in any number of dimensions. We use the same ansatz $\mathbf{A} = \mathbf{B}\,\nabla Z$, solve the Lorenz equation as earlier, and show that the Proca equation is separable. The only difference is that a separation constant, that for Maxwell eqns vanishes, in the Proca case takes the value $\sim \mathrm{m}^2$.

BIG PICTURE



The principal tensor, which exists in higher dimensional black holes, provides us with powerful tools that allow us to study their properties. But why at all the Nature 'decided' to give us such a gift?

“Black Holes, Hidden Symmetry and Complete Integrability”

V.F., Pavel Krtous and David Kubiznak

Prepared for Living Reviews in Relativity, arXiv:1705.05482 (2017)

Ultralight weakly-coupled new particles are a feature of many beyond Standard Model (SM) scenarious. Their couplings to the SM particles are not known. However, They have universal coupling to gravity. Superradiance results in two effects:

(i) spinning down BH rotation, and

(ii) emission of gravitational waves.

For stellar mass BHs: $10^{-13} eV \leq m \leq 10^{-12} eV$;

For SMBHs: $10^{-19} eV \leq m \leq 10^{-13} eV$.

$$\Phi \sim \exp(-i\omega t + im\phi)Y(r, y);$$

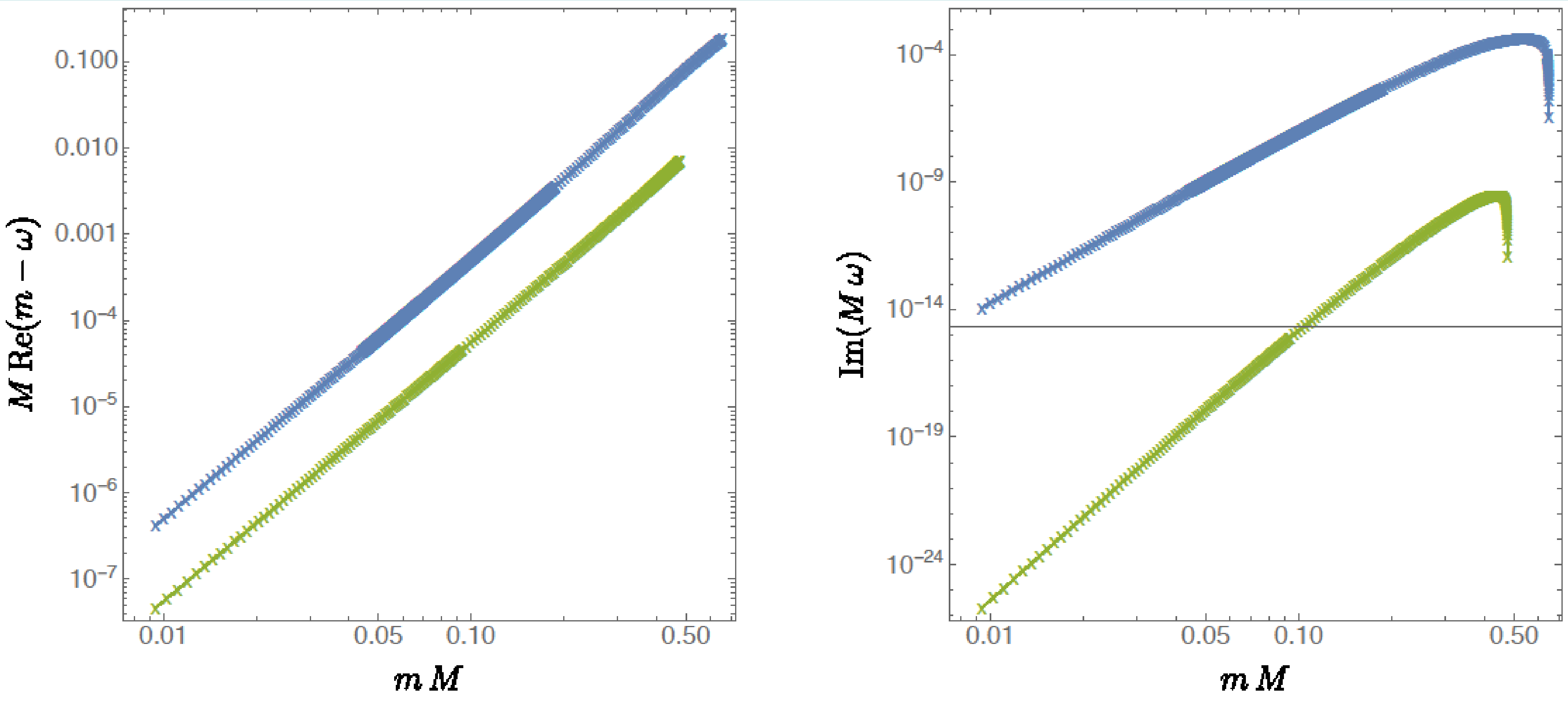
Non-rotating compact object:

$$\omega \sim \mu(1\text{-}\frac{(\mu M)^2}{n^2}); \quad \Im(\omega) = 0;$$

Non-rotating BH: $\Im(\omega) < 0;$

Rotating BHs: $\Im(\omega) \sim Q\,(m\Omega - \omega);$

$$Q \sim (2\mu M)^{2j+2l+5}.$$



0.100
0.010
0.001
10^-4
10^-5
10^-6
10^-7
M Re(m − ω)
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0.05
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m M
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10^-19
10^-24
Im(M ω)
0.01
0.05
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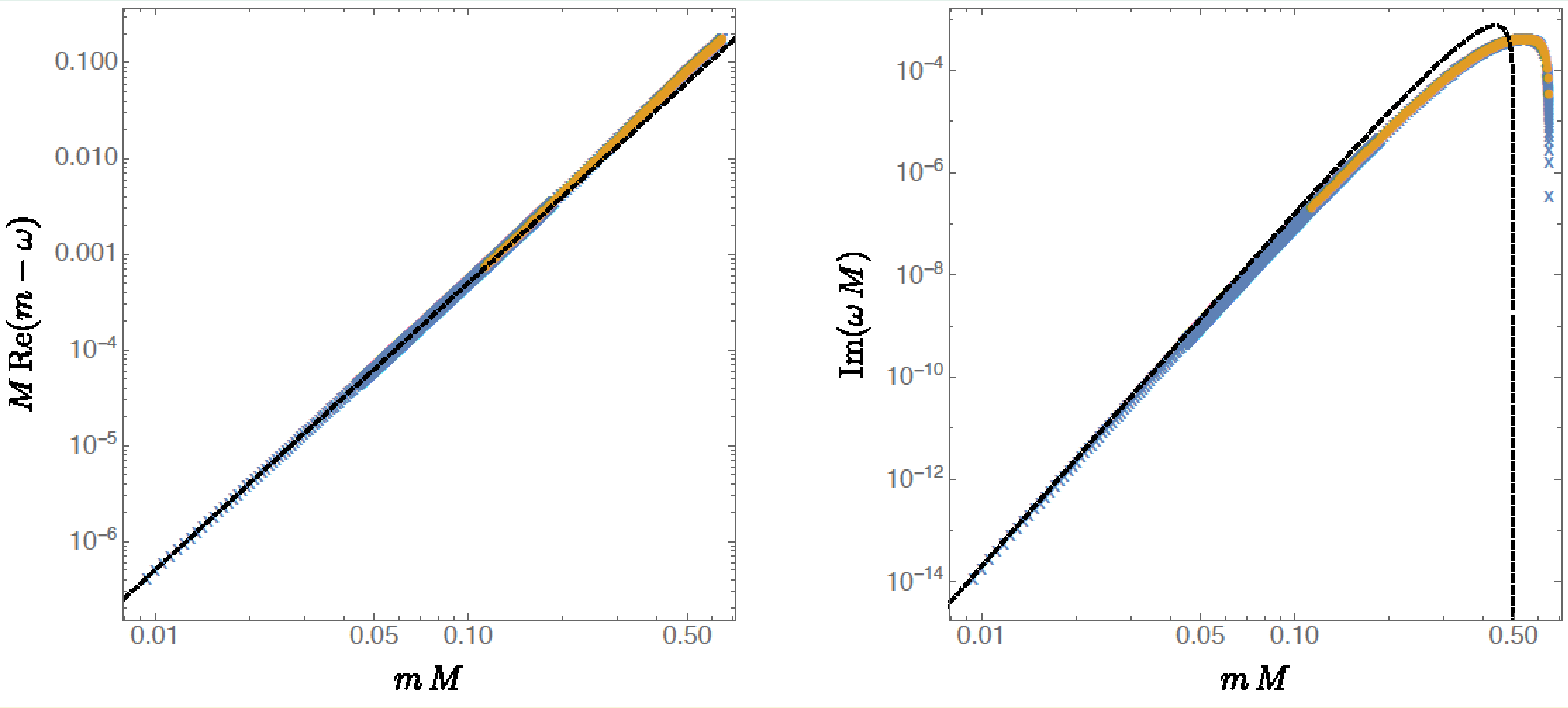
0.50
m M



0.100
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10^{-4}
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$M\,\mathrm{Re}(m-\omega)$
0.01
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$m\,M$
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10^{-12}
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$\mathrm{Im}(\omega\,M)$
0.01
0.05
0.10
0.50
$m\,M$

Higher Dimensional Kerr-NUT-(A)dS Black Holes

Tangherlini '63 metric (HD Schw. analogue)

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Myers & Perry '86 metric (HD Kerr analogue)

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Kerr - NUT - AdS (Chen, Lu, and Pope '06)

"General Kerr-NUT-AdS metrics in all dimensions“, Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$R_{ab} = \frac{2}{D-2}\Lambda g_{ab}, \quad D = 2n+\varepsilon$$

Λ, M – mass, a_k – $(n-1+\varepsilon)$ rotation parameters,

N_α – $(n-1-\varepsilon)$ `NUT' parameters

Total # of parameters is $2n-1$

Operations with forms

(1) External product: $\alpha_q \wedge \beta_p = (\alpha \wedge \beta)_{q+p}$;

(2) Hodge duality: $*(\alpha_p) = (*\alpha)_{D-p}$

$$(*\alpha)_{a_{p+1}\dots a_D} = \frac{1}{p!}\alpha^{a_1\dots a_p} e_{a_1\dots a_p a_{p+1}\dots a_D},$$

e is totally antisymmetric tensor;

(3) External derivative: $d(\alpha_q) = (d\alpha)_{q+1}$;

(4) Co-derivative: $\delta(\alpha_p) = (\delta\alpha)_{p-1} = (-1)^p \varepsilon_{p-1} * d * \alpha_p$;

(5) Contraction: $X \cdot \omega \Leftrightarrow (X \cdot \omega)_{a_2\dots a_p} = X^a \omega_{a a_2\dots a_p}$.