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Tangent lifts
of bi-Hamiltonian structures

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A Poisson manifold \((M, \{\cdot, \cdot\})\) is a smooth manifold \(M\) (equipped with a Poisson structure) with a fixed bilinear and antisymmetric mapping \(\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)\), which satisfies Leibniz rule and Jacobi identity.

\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,
\]

\[
\{f, gh\} = \{f, g\}h + g\{f, h\},
\]

where \(f, g, h \in C^\infty(M)\).

Poisson bracket can be written in terms of Poisson tensor \((\pi \in \Gamma^\infty (\Lambda^2 TM)\) such that \([\pi, \pi]_{S-N} = 0\) as follows

\[
\{f, g\} = \pi(df, dg).
\]
In the local coordinates $x_1, x_2, \ldots, x_N$ on $M$

\[ \{f, g\} = \sum_{i,j=1}^{N} \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}. \]

Components of Poisson tensor are given by the formula

\[ \pi_{ij}(x) = \{x_i, x_j\} \]

and satisfy

- $\pi_{ij} = -\pi_{ji},$
- $\frac{\partial \pi_{ij}}{\partial x_s}\pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s}\pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s}\pi_{si} = 0.$

Choosing the function $H$ as a Hamiltonian we can define a dynamics on $M$ using Hamilton equations

\[ \frac{dx_i}{dt} = \{x_i, H\}, \quad i = 1, 2, \ldots, N, \]
\[ \frac{dx}{dt} = \pi \nabla H, \]
Bi-Hamiltonian structures

Let $M$ be a manifold with two non-proportional Poisson brackets $\{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2$. If their linear combination $\alpha\{\cdot, \cdot\}_1 + \beta\{\cdot, \cdot\}_2$, $\alpha, \beta \in \mathbb{R}$, is also a Poisson bracket, we say that the brackets are compatible and we call $M$ the bi-Hamiltonian manifold. By analogy we will say that two Poisson tensors $\pi_1$ and $\pi_2$ are compatible if their Schouten–Nijenhuis bracket vanishes

$$[\pi_1, \pi_2]_{S-N} = 0.$$
Example – Bi-Hamiltonian structure related to \( \mathfrak{so}(3) \)

Let us consider the Lie algebra \( \mathfrak{so}(3) \) of skew-symmetric matrices. We will now construct two Lie brackets on \( \mathfrak{so}(3) \) given by two choices of the matrix \( S \)

\[
[A, B] = AB - BA, \quad [A, B]_S = ASB - BSA,
\]

where \( S = \text{diag}(s_1, s_2, s_3) \). We define the Lie–Poisson bracket

\[
\{f, g\}_1(\rho) = \langle \rho, [df(\rho), dg(\rho)] \rangle = \frac{1}{2} \text{tr} \left( \rho [df(\rho), dg(\rho)] \right),
\]

\[
\{f, g\}_2(\rho) = \langle \rho, [df(\rho), dg(\rho)]_S \rangle = \frac{1}{2} \text{tr} \left( \rho [df(\rho), dg(\rho)]_S \right),
\]

The Poisson tensors can be written in the form

\[
\pi_1(X) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad \pi_2(X) = \begin{pmatrix} 0 & -s_3x_3 & s_2x_2 \\ s_3x_3 & 0 & -s_1x_1 \\ -s_2x_2 & s_1x_1 & 0 \end{pmatrix}.
\]
In this case, the Casimirs for these structures assume the following form

\[ c_1(X) = x_1^2 + x_2^2 + x_3^2, \quad c_2(X) = s_1 x_1^2 + s_2 x_2^2 + s_3 x_3^2. \]

Choosing as the Hamiltonian the Casimir \( c_2 \) we obtain Euler’s equation, which describes the rotation of a rigid body

\[
\frac{d\vec{x}}{dt} = \{ c_2, \vec{x} \}_1 = \{ c_1, \vec{x} \}_2 = 2 (S_2 \vec{x}) \times \vec{x},
\]

where \( \vec{x} = (x_1, x_2, x_2) \) and \( S_2 = \text{diag} (s_1, s_2, s_3) \).
Lie Algebroid

Definition

Let $M$ be a manifold. A Lie algebroid on $M$ is a vector bundle $A \to M$, together with a vector bundle map $a : A \to TM$, called the anchor of a Lie algebroid $A$, and a bracket $[\cdot, \cdot]_A : \Gamma_A \times \Gamma_A \to \Gamma_A$ which is $\mathbb{R}$-bilinear and alternating, satisfies the Jacobi identity ($\Gamma_A$ is a Lie algebra), and is such that

\begin{align}
[X, fY]_A &= f[X, Y]_A + a(X)(f)Y, \\
(a([X, Y]_A) &= [a(X), a(Y)],
\end{align}

for all $X, Y \in \Gamma_A$, $f \in C^\infty(M)$. The manifold $M$ is called the base of a Lie algebroid $A$. 
Let \((M, \{.,.\})\) be a Poisson manifold, then its cotangent bundle \(T^*M \to M\) possesses a Lie algebroid structure given by

\[
a(df) := \{f,.\}
\]

\[
[df, dg]_{T^*M} := d\{f, g\},
\]

where \(f, g \in C^\infty(M)\).
If \((A \to M, [\cdot, \cdot]_A, a)\) is a Lie algebroid then on the total space \(A^*\) of dual bundle \(A^* \to M\) there exists a Poisson structure given by

\[
\{f \circ q, g \circ q\} = 0,
\]

\[
\{l_X, g \circ q\} = a(X)(g) \circ q \tag{3}
\]

\[
\{l_X, l_Y\} = l_{[X,Y]}_A,
\]

where \(X, Y \in \Gamma^\infty(A),\) \(l_X(v) = \langle v, X(q(v)) \rangle,\) \(v \in A^*\) and \(f, g \in C^\infty(M).\)
If \((M, \{, \})\) is a Poisson manifold, then the manifold \(TM\) possesses a Poisson structure given by

\[
\{ f \circ q, g \circ q \}\_{TM} = 0,
\]

\[
\{l_{df}, g \circ q \}\_{TM} = \{f, g\} \circ q
\]

\[
\{l_{df}, l_{dg} \}\_{TM} = l_{d\{f,g\}},
\]

where \(l_{df}(v) = \langle v, df(q(v)) \rangle\), \(v \in TM\) and \(f, g \in C^\infty(M)\).
Corollary

Let \((M, \pi)\) be a Poisson manifold and let \(x = (x_1, \ldots, x_N)\) be a system of local coordinates on \(M\). Then the Poisson tensor \(\pi_{TM}\) on the manifold \(TM\) associated with \(\pi\) has the form

\[
\pi_{TM}(x, y) = \begin{pmatrix}
0 \\
\pi(x_1, \ldots, x_N) \\
\pi(x_1, \ldots, x_N) \sum_{s=1}^{N} \frac{\partial \pi}{\partial x_s}(x_1, \ldots, x_N)y_s
\end{pmatrix},
\]

in the system of local coordinates \((x, y) = (x_1, \ldots, x_N, y_1, \ldots, y_N)\) on \(TM\).
Lifting of Casimir functions from $M$ to $TM$

**Theorem**

Let $c_1, \ldots, c_r$, where $r = \dim M - \text{rank} \pi$, be Casimir functions for the Poisson structure $\pi$, then the functions

$$c_i \circ q \quad \text{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \quad i = 1, \ldots r,$$

are the Casimir functions for the Poisson tensor $\pi_{TM}$. 
Lifting of functions in involution from $M$ to $TM$

**Theorem**

Let functions $\{H_i\}_{i=1}^k$ be in involution with respect to the Poisson bracket generated by $\pi$, then the functions

$$\{H_i \circ q, \{dH_i = \sum_{s=1}^N \frac{\partial H_i}{\partial x_s}(x)y_s\}_{i=1}^k,$$

are in involution with respect to the Poisson tensor $\pi_{TM}$.

**Theorem**

If $(M, \pi_1, \pi_2)$ is a bi-Hamilton manifold then $(TM, \pi_{1,TM}, \pi_{2,TM})$ is a bi-Hamilton manifold.
In the case of a linear Poisson structure, we have additionally a Lie-Poisson structure on $TM$.

**Theorem**

Let $\pi$ be the Lie-Poisson structure on $\mathfrak{g}^*$. Then the tensor

$$\tilde{\pi}_{T\mathfrak{g}^*}(x, y) = \left( \begin{array}{c|c} \lambda \pi(y) & \pi(x) \\ \hline \pi(x) & \pi(y) \end{array} \right)$$

gives the Poisson structure on $T\mathfrak{g}^*$ for any $\lambda \in \mathbb{R}$.

**Theorem**

Let $c_1, \ldots, c_r$, where $r = \dim M - \text{rank} \pi$, be Casimir functions for the Poisson structure $\pi$ with $\lambda \neq 0$, then the functions

$$c_i(t) + c_i(w) \quad c_i(t) - c_i(w), \quad i = 1, \ldots r,$$

where $t = (x_1 - \sqrt{\lambda}y_1, \ldots, x_N - \sqrt{\lambda}y_N)$,

$w = (x_1 + \sqrt{\lambda}y_1, \ldots, x_N + \sqrt{\lambda}y_N)$, are the Casimir functions.
The Poisson structures on $T\mathfrak{so}(3)$ are given by tensors

$$
\pi_{1,TM}(X,Y) = \begin{pmatrix}
0 & 0 & 0 & 0 & -x_3 & x_2 \\
0 & 0 & 0 & x_3 & 0 & -x_1 \\
0 & 0 & 0 & -x_2 & x_1 & 0 \\
0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\
x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\
-x_2 & x_1 & 0 & -y_2 & y_1 & 0
\end{pmatrix}.
$$

Moreover the Casimirs are given by

$$
c_1(X) = x_1^2 + x_2^2 + x_3^2, \quad \frac{1}{2} l_{dc_1} = x_1 y_1 + x_2 y_2 + x_3 y_3.
$$

In this case we recognize the Lie-Poisson structure of $\mathfrak{e}(3) \cong T\mathfrak{so}(3)$. 

Tangent lifts of bi-Hamiltonian structures
We have another Poisson structure on $T\mathfrak{so}(3)$

\[
\tilde{\pi}_{1,TM}(X, Y) = \begin{pmatrix}
0 & -y_3 & y_2 & 0 & -x_3 & x_2 \\
y_3 & 0 & -y_1 & x_3 & 0 & -x_1 \\
-y_2 & y_1 & 0 & -x_2 & x_1 & 0 \\
0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\
x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\
-x_2 & x_1 & 0 & -y_2 & y_1 & 0 \\
\end{pmatrix}.
\]

In this case, we recognize the Lie-Poisson structure of $\mathfrak{so}(4) \cong T\mathfrak{so}(3)$. The Casimir functions now are given by the formulas

\[
c_1(X + Y) + c_1(X - Y) = 2 \left( x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \right),
\]
\[
c_1(X + Y) - c_1(X - Y) = 4 \left( x_1 y_1 + x_2 y_2 + x_3 y_3 \right).
\]
Lifting of a bi-Hamiltonian structure from $M$ to $TM$ (Main results)

**Theorem**

If $(M, \{, \}_1, \{, \}_2)$ is a bi-Hamiltonian manifold, then for any $\lambda \in \mathbb{R}$ its tangent bundle $TM$ possesses a Poisson structure $\{, \}_{TM,\lambda}$ given by

$$\{f \circ q, g \circ q\}_{TM,\lambda} = 0,$$

$$\{l_{df}, g \circ q\}_{TM,\lambda} = \{f, g\}_1 \circ q$$  \hspace{1cm} (6)

$$\{l_{df}, l_{dg}\}_{TM,\lambda} = l_d\{f,g\}_1 + \lambda\{f,g\}_2 \circ q,$$

where $l_{df}(v) = \langle v, df(q(v)) \rangle, \ v \in TM$ and $f, g \in C^\infty(M)$. 

**Tangent lifts of bi-Hamiltonian structures**

Corollary

Let \((M, \pi_1, \pi_2)\) be a bi-Hamiltonian manifold and let \(x = (x_1, \ldots, x_N)\) be a system of local coordinates on \(M\). Then the Poisson tensor \(\pi_{TM,\lambda}\) related to \((M, \pi_1, \pi_2)\) takes form

\[
\pi_{TM,\lambda}(x, y) = \begin{pmatrix}
0 \\
\pi_1(x) \\
\sum_{s=1}^{N} \frac{\partial \pi_1}{\partial x_s}(x) y_s + \lambda \pi_2(x)
\end{pmatrix},
\]

in the system of local coordinates \((x, y) = (x_1, \ldots, x_N, y_1, \ldots, y_N)\) on \(TM\).
Lifting of Casimir functions from $M$ to $TM$

**Theorem**

Let $c_1, \ldots, c_r$, where $r = \dim M - \text{rank} \pi$, be Casimir functions for the Poisson structure $\pi_1$ and functions $f_i^\lambda$, $i = 1, \ldots, r$, satisfy the conditions $\{f_i^\lambda, x_j\}_1 = \{x_j, c_i\}_2$, for $j = 1, \ldots, n$, then the functions

$$c_i \circ q \quad \text{and} \quad \tilde{c}_i = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(x)y_s + \lambda f_i^\lambda(x), \quad i = 1, \ldots r,$$

are the Casimir functions for the Poisson tensor $\pi_{TM,\lambda}$.

**Theorem**

If the functions $\{H_i\}$ are in involution with respect to the Poisson tensor $\pi$ then the functions $\{H_i \circ q, \tilde{H}_i = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(x)y_s\}$ are in involution with respect to the Poisson tensor $\pi_{TM,\lambda}$.
The Hamiltonian

\[ H = \sum_{i \in \mathbb{Z}} \left( \frac{1}{2} p_i^2 + e^{q_{i-1} - q_i} \right) . \]

Hamilton’s equations

\[
\begin{aligned}
\dot{q}_i &= \{q_i, H\} = p_i \\
\dot{p}_i &= \{p_i, H\} = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}}.
\end{aligned}
\]

Under Flaschka’s transformation

\[ a_i = \frac{1}{2} e^{\frac{(q_{i-1} - q_i)}{2}}, \quad b_i = -\frac{1}{2} p_{i-1} \]

the system transforms to

\[
\begin{aligned}
\frac{da_i}{dt} &= a_i (b_{i+1} - b_i) , \\
\frac{db_i}{dt} &= 2 \left( a_i^2 - a_{i-1}^2 \right).
\end{aligned}
\]
The Toda lattice is equivalent to the Lax equation

\[ \frac{dL}{dt} = [A, L], \]

where

\[ Lf_i = a_i f_{i+1} + b_if_i + a_{i-1}f_{i-1}, \]

\[ Af_i = a_i f_{i+1} - a_{i-1}f_{i-1} \]

are linear operators in the Hilbert space of square summable sequences \( l^2(\mathbb{Z}) \).
The Toda lattice is a bi-Hamiltonian system. There exist another Poisson bracket, which we denote by $\pi_2$, and another function $H_1$, which will play the role of the Hamiltonian for the $\pi_2$ bracket, such that $\pi_1 + \pi_2$ is Poisson tensor and $\pi_1 \nabla H = \pi_2 \nabla H_1$ ($H = \sum_i \left(2b_i^2 + 4a_i^2\right)$). The Poisson tensor $\pi_1$ is given by the relations

$$8\{a_i, b_i\}_1 = -a_i, \quad 8\{a_i, b_{i+1}\}_1 = a_i.$$ 

For the Toda lattice the $\pi_2$ bracket (which appeared in a paper of M. Adler) is quadratic in the variables $b_i, a_i$ and it is given by the relations

$$\{a_i, a_{i+1}\}_2 = \frac{1}{2}a_ia_{i+1}, \quad \{a_i, b_i\}_2 = -a_ib_i,$$

$$\{a_i, b_{i+1}\}_2 = a_ib_{i+1}, \quad \{b_i, b_{i+1}\}_2 = 2a_i^2$$

and all other brackets are zero.
Example - Extended Toda Lattice

Functions $H_k = Tr L^k$ are the functions in involutions with respect to the both brackets. The above functions for $k = 1, 2, 3$ have the expressions

$$H_1 = tr L = \sum_{i \in \mathbb{Z}} b_i, \quad H_2 = 2H = tr L^2 = \sum_{i \in \mathbb{Z}} (b_i^2 + 2a_i^2), \quad (7)$$

$$H_3 = tr L^3 = \sum_{i \in \mathbb{Z}} (b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}).$$

Now deformed tangent Poisson structure $\pi_{TM,\lambda}$ in local coordinates $a_i, b_i, n_i, m_i, i \in \mathbb{Z}$, is given by the relation

$$\{a_i, m_i\}_{TM,\lambda} = -\frac{1}{4}a_i, \quad \{a_i, m_{i+1}\}_{TM,\lambda} = \frac{1}{4}a_i, \quad (8)$$

$$\{b_i, n_i\}_{TM,\lambda} = \frac{1}{4}a_i, \quad \{b_{i+1}, n_i\}_{TM,\lambda} = -\frac{1}{4}a_i, \quad (9)$$

$$\{n_i, n_{i+1}\}_{TM,\lambda} = \frac{\lambda}{2}a_i a_{i+1}, \quad \{n_i, m_i\}_{TM,\lambda} = -\frac{1}{4}n_i - \lambda a_i b_i, \quad (10)$$

Tangent lifts of bi-Hamiltonian structures
From the last theorem we transform the functions $H_k = TrL^k$ into the functions $H_k \circ q^*_M = TrL^k \circ q^*_M$ and

$\tilde{H}_k = \sum_{s=1}^{N} \left( \frac{\partial H_k}{\partial a_s} n_s + \frac{\partial H_k}{\partial b_s} m_s \right)$, i.e.

$H_1 = \sum_{i \in \mathbb{Z}} b_i, \quad \tilde{H}_1 = \sum_{i \in \mathbb{Z}} m_i,$

$H_2 = \sum_{i \in \mathbb{Z}} (b_i^2 + 2a_i^2), \quad \tilde{H}_2 = \sum_{i \in \mathbb{Z}} (2b_i m_i + 4a_i n_i),$ 

$H_3 = \sum_{i \in \mathbb{Z}} (b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}), \quad \tilde{H}_3 = \sum_{i \in \mathbb{Z}} (3b_i^2 m_i + 3a_i^2 m_i + 3a_i^2 m_i + 6a_i b_i n_i + 6a_i b_{i+1} n_i),$ 

\ldots 

Tangent lifts of bi-Hamiltonian structures
Now if we take as the Hamiltonian
\[
H = \alpha H_2 + \beta \tilde{H}_2 = \sum_{i \in \mathbb{Z}} \left( \alpha b_i^2 + 2\alpha a_i^2 + 2\beta b_i m_i + 4\beta a_i n_i \right) \tag{12}
\]
then Hamilton’s equations are in the form
\[
\frac{d a_i}{dt} = \frac{1}{2} \beta a_i (b_{i+1} - b_i),
\]
\[
\frac{d b_i}{dt} = \beta (a_i^2 - a_{i-1}^2),
\]
\[
\frac{d n_i}{dt} = \frac{1}{2} \alpha a_i (b_{i+1} - b_i) + \frac{1}{2} \beta a_i (m_{i+1} - m_i) + \frac{1}{2} \beta n_i (b_{i+1} - b_i) + 2\beta \lambda a_i (a_{i+1}^2 - a_{i-1}^2 - b_i^2 + b_{i+1}^2),
\]
\[
\frac{d m_i}{dt} = \alpha (a_i^2 - a_{i-1}^2) + 2\beta (a_i n_i - a_{i-1} n_{i-1}) + 4\beta \lambda \left( a_i^2 b_{i+1} + a_i^2 b_i - a_{i-1}^2 b_i - a_{i-1}^2 b_{i-1} \right).
\]
We can interpret these equations as an extension of the Toda lattice. It is the integrable system, where the integrals of motions are given by formulas (11). If we put \(\alpha = \lambda = 0, \beta = 2\) and we take \(n_i = m_i = 0\) then we observe that we reduce it to Toda lattice.
Thank you for your attention