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Tangent lifts of bi-Hamiltonian structures

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Poisson manifold $(M, \{\cdot,\cdot\})$

Definition

A Poisson manifold $(M, \{\cdot,\cdot\})$ is a smooth manifold M (equipped with a Poisson structure) with a fixed bilinear and antisymmetric mapping $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which satisfies Leibniz rule and Jacobi identity.

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

$$\{f,gh\} = \{f,g\}h + g\{f,h\},$$

where $f,g,h \in C^\infty(M)$.

Poisson bracket can be written in terms of **Poisson tensor** ($\pi \in \Gamma^\infty\left(\bigwedge^2 TM\right)$ such that $[\pi,\pi]_{S-N} = 0$) as follows

$$\{f,g\} = \pi(df, dg).$$

Poisson tensor, Hamilton's equations

In the local coordinates $x_1, x_2, \ldots, x_N$ on M

$$\{f, g\} = \sum_{i,j=1}^{N} \pi_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Components of Poisson tensor are given by the formula

$$\pi_{ij}(x) = \{x_i, x_j\}$$

and satisfy

- $\pi_{ij} = -\pi_{ji}$,
- $\frac{\partial \pi_{ij}}{\partial x_s} \pi_{sk} + \frac{\partial \pi_{ki}}{\partial x_s} \pi_{sj} + \frac{\partial \pi_{jk}}{\partial x_s} \pi_{si} = 0.$

Choosing the function H as a Hamiltonian we can define a dynamics on M using Hamilton equations

$$\frac{dx_i}{dt} = \{x_i, H\}, \qquad i = 1, 2, \ldots, N,$$

$$\frac{dx}{dt} = \pi \nabla H,$$

Bi-Hamiltonian structures

Let M be a manifold with two non-proportional Poisson brackets $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$. If their linear combination $\alpha\{\cdot,\cdot\}_1+\beta\{\cdot,\cdot\}_2$, $\alpha,\beta\in\mathbb{R}$, is also a Poisson bracket, we say that the brackets are compatible and we call M the bi-Hamiltonian manifold.
By analogy we will say that two Poisson tensors π_1 and π_2 are compatible if their Schouten–Nijenhuis bracket vanishes

$$[\pi_1,\pi_2]_{S-N}=0.$$

$$\frac{\partial\pi_1^{ij}}{\partial x^s}\pi_2^{sk}+\frac{\partial\pi_2^{ij}}{\partial x^s}\pi_1^{sk}+\frac{\partial\pi_1^{ki}}{\partial x^s}\pi_2^{sj}+\frac{\partial\pi_2^{ki}}{\partial x^s}\pi_1^{sj}+\frac{\partial\pi_1^{jk}}{\partial x^s}\pi_2^{si}+\frac{\partial\pi_2^{jk}}{\partial x^s}\pi_1^{si}=0.$$

Example– Bi-Hamiltonian structure related to so(3)

Let us consider the Lie algebra $\mathfrak{so}(3)$ of skew-symmetric matrices. We will now construct two Lie brackets on $\mathfrak{so}(3)$ given by two choices of the matrix S

$$[A, B] = AB - BA, \quad [A, B]_S = ASB - BSA,$$

where $S = \mathrm{diag}(s_1, s_2, s_3)$. We define the Lie–Poisson bracket

$$\{f, g\}_1(\rho) = \langle \rho, [df(\rho), dg(\rho)] \rangle = \frac{1}{2}\mathrm{tr}\left(\rho[df(\rho), dg(\rho)]\right),$$

$$\{f, g\}_2(\rho) = \langle \rho, [df(\rho), dg(\rho)]_S \rangle = \frac{1}{2}\mathrm{tr}\left(\rho[df(\rho), dg(\rho)]_S\right),$$

The Poisson tensors can be written in the form

$$\pi_1(X) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \pi_2(X) = \begin{pmatrix} 0 & -s_3x_3 & s_2x_2 \\ s_3x_3 & 0 & -s_1x_1 \\ -s_2x_2 & s_1x_1 & 0 \end{pmatrix}.$$

In this case, the Casimirs for these structures assume the following form

$$c_1(X) = x_1^2 + x_2^2 + x_3^2, \quad c_2(X) = s_1 x_1^2 + s_2 x_2^2 + s_3 x_3^2.$$

Choosing as the Hamiltonian the Casimir c_2 we obtain Euler's equation, which describes the rotation of a rigid body

$$\frac{d\vec{x}}{dt} = \{c_2, \vec{x}\}_1 = \{c_1, \vec{x}\}_2 = 2\,(S_2\vec{x}) \times \vec{x},$$

where $\vec{x} = (x_1, x_2, x_2)$ and $S_2 = \mathrm{diag}\,(s_1, s_2, s_3)$.

Lie Algebroid

Definition

Let M be a manifold. A Lie algebroid on M is a vector bundle $A \to M$, together with a vector bundle map $a : A \longrightarrow TM$, called the anchor of a Lie algebroid A, and a bracket $[\cdot,\cdot]_A : \Gamma A \times \Gamma A \longrightarrow \Gamma A$ which is $\mathbb{R}$–bilinear and alternating, satisfies the Jacobi identity (ΓA is a Lie algebra), and is such that

$$[X, fY]_A = f[X,Y]_A + a(X)(f)Y, \tag{1}$$

$$a\left([X,Y]_A\right) = [a(X), a(Y)], \tag{2}$$

for all $X, Y \in \Gamma A$, $f \in C^\infty(M)$. The manifold M is called the base of a Lie algebroid A.

Example

Let $(M, \{.,.\})$ be a Poisson manifold, then its cotangent bundle $T^*M \to M$ possesses a Lie algebroid structure given by

$$a(df) := \{f, .\}$$

$$[df, dg]_{T^*M} := d\{f, g\},$$

where $f, g \in C^\infty(M)$.

Linear Fiber-wise Poisson Structure

If $(A \to M, [\cdot,\cdot]_A, a)$ is a Lie algebroid then on the total space A^* of dual bundle $A^* \xrightarrow{q} M$ there exists a Poisson structure given by

$$\{f \circ q, g \circ q\} = 0,$$

$$\{l_X, g \circ q\} = a(X)(g) \circ q \tag{3}$$

$$\{l_X, l_Y\} = l_{[X,Y]_A},$$

where $X, Y \in \Gamma^\infty(A)$, $l_X(v) = \langle v, X(q(v))\rangle$, $v \in A^*$ and $f, g \in C^\infty(M)$.

Lifting of a Hamiltonian structure from M to TM

Theorem

If $(M, \{,\})$ is a Poisson manifold, then the manifold TM possesses a Poisson structure given by

$$\{f \circ q, g \circ q\}_{TM} = 0,$$

$$\{l_{df}, g \circ q\}_{TM} = \{f, g\} \circ q \tag{4}$$

$$\{l_{df}, l_{dg}\}_{TM} = l_{d\{f,g\}},$$

where $l_{df}(v) = \langle v, df(q(v))\rangle$, $v \in TM$ and $f, g \in C^\infty(M)$.

Corollary

Let (M,π) *be a Poisson manifold and let* $\mathbf{x} = (x_1, \ldots, x_N)$ *be a system of local coordinates on* M. *Then the Poisson tensor* π_{TM} *on the manifold* TM *associated with* π *has the form*

$$\pi_{TM}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi(x_1,\ldots,x_N) \\ \hline \pi(x_1,\ldots,x_N) & \sum_{s=1}^{N} \dfrac{\partial \pi}{\partial x_s}(x_1,\ldots,x_N) y_s \end{array}\right),$$

in the system of local coordinates $(\mathbf{x},\mathbf{y}) = (x_1,\ldots,x_N,y_1,\ldots,y_N)$ *on* TM.

Lifting of Casimir functions from M to TM

Theorem

Let $c_1, \dots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the the Poisson structure π, then the functions

$$c_i \circ q \quad \text{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s} y_s, \quad i = 1, \dots r,$$

are the Casimir functions for the Poisson tensor π_{TM}.

Lifting of functions in involution from M to TM

Theorem

Let functions $\{H_i\}_{i=1}^k$ be in involution with respect to the Poisson bracket generated by π, then the functions

$$\{H_i \circ q,\ l_{dH_i} = \sum_{s=1}^{N} \frac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s\}_{i=1}^k, \tag{5}$$

are in involution with respect to the Poisson tensor π_{TM}.

Theorem

If (M, π_1, π_2) is a bi-Hamilton manifold then $(TM, \pi_{1,TM}, \pi_{2,TM})$ is a bi-Hamilton manifold.

In the case of a linear Poisson structure, we have additionally a Lie-Poisson structure on TM.

Theorem

Let π be the Lie-Poisson structure on $\mathfrak{g}^*$. Then the tensor

$$\tilde{\pi}_{T\mathfrak{g}^*}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} \lambda\pi(\mathbf{y}) & \pi(\mathbf{x}) \\ \hline \pi(\mathbf{x}) & \pi(\mathbf{y}) \end{array}\right)$$

gives the Poisson structure on $T\mathfrak{g}^*$ for any $\lambda \in \mathbb{R}$.

Theorem

Let $c_1, \ldots, c_r$, where $r = \dim M - \operatorname{rank}\pi$, be Casimir functions for the Poisson structure π with $\lambda \neq 0$, then the functions

$$c_i(\mathbf{t}) + c_i(\mathbf{w}) \quad c_i(\mathbf{t}) - c_i(\mathbf{w}), \quad i = 1, \ldots r,$$

where $\mathbf{t} = \left(x_1 - \sqrt{\lambda}y_1, \ldots, x_N - \sqrt{\lambda}y_N\right)$,
$\mathbf{w} = \left(x_1 + \sqrt{\lambda}y_1, \ldots, x_N + \sqrt{\lambda}y_N\right)$, are the Casimir functions.

Example

The Poisson structures on $T\mathfrak{so}(3)$ are given by tensors

$$\pi_{1,TM}(X,Y) = \begin{pmatrix} 0 & 0 & 0 & 0 & -x_3 & x_2 \\ 0 & 0 & 0 & x_3 & 0 & -x_1 \\ 0 & 0 & 0 & -x_2 & x_1 & 0 \\ 0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\ x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\ -x_2 & x_1 & 0 & -y_2 & y_1 & 0 \end{pmatrix}.$$

Moreover the Casimirs are given by

$$c_1(X) = x_1^2 + x_2^2 + x_3^2, \qquad \frac{1}{2} l_{dc_1} = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

In this case we recognize the Lie-Poisson structure of $\mathfrak{e}(3) \cong T\mathfrak{so}(3)$.

We have another Poisson structure on $T\mathfrak{so}(3)$

$$\tilde{\pi}_{1,TM}(X,Y) = \begin{pmatrix} 0 & -y_3 & y_2 & 0 & -x_3 & x_2 \\ y_3 & 0 & -y_1 & x_3 & 0 & -x_1 \\ -y_2 & y_1 & 0 & -x_2 & x_1 & 0 \\ 0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\ x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\ -x_2 & x_1 & 0 & -y_2 & y_1 & 0 \end{pmatrix}.$$

In this case, we recognize the Lie-Poisson structure of $\mathfrak{so}(4) \cong T\mathfrak{so}(3)$. The Casimir functions now are given by the formulas

$$c_1(X+Y) + c_1(X-Y) = 2\left(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2\right),$$
$$c_1(X+Y) - c_1(X-Y) = 4\left(x_1 y_1 + x_2 y_2 + x_3 y_3\right).$$

Lifting of a bi-Hamiltonian structure from M to TM (Main results)

Theorem

If $(M, \{,\}_1, \{,\}_2)$ is a bi-Hamiltonian manifold, then for any $\lambda \in \mathbb{R}$ its tangent bundle TM possesses a Poisson structure $\{,\}_{TM,\lambda}$ given by

$$\{f \circ q, g \circ q,\}_{TM,\lambda} = 0,$$

$$\{l_{df}, g \circ q\}_{TM,\lambda} = \{f, g\}_1 \circ q \tag{6}$$

$$\{l_{df}, l_{dg}\}_{TM,\lambda} = l_{d\{f,g\}_1} + \lambda\{f, g\}_2 \circ q,$$

where $l_{df}(v) = \langle v, df(q(v))\rangle,\ v \in TM$ and $f, g \in C^\infty(M)$.

Lifting of a bi-Hamiltonian structure from M to TM

Corollary

Let (M, π_1, π_2) *be a bi-Hamiltonian manifold and let* $\mathbf{x} = (x_1, \dots, x_N)$ *be a system of local coordinates on* M*. Then the Poisson tensor* $\pi_{TM,\lambda}$ *related to* (M, π_1, π_2) *takes form*

$$\pi_{TM,\lambda}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{c|c} 0 & \pi_1(\mathbf{x}) \\ \hline \pi_1(\mathbf{x}) & \sum_{s=1}^{N} \frac{\partial \pi_1}{\partial x_s}(\mathbf{x}) y_s + \lambda \pi_2(\mathbf{x}) \end{array}\right),$$

in the system of local coordinates $(\mathbf{x},\mathbf{y}) = (x_1, \dots, x_N, y_1, \dots, y_N)$ *on* TM*.*

Lifting of Casimir functions from M to TM

Theorem

Let $c_1, \dots, c_r$, where $r = \dim M - \operatorname{rank} \pi$, be Casimir functions for the Poisson structure π_1 and functions f_i^λ, $i = 1, \dots, r$, satisfy the conditions $\{f_i^\lambda, x_j\}_1 = \{x_j, c_i\}_2, \quad$ for $\quad j = 1, \cdots, n$, then the functions

$$c_i \circ q \quad \text{and} \quad \tilde{c}_i = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(\mathbf{x}) y_s + \lambda f_i^\lambda(\mathbf{x}), \quad i = 1, \dots r,$$

are the Casimir functions for the Poisson tensor $\pi_{TM,\lambda}$.

Theorem

If the functions $\{H_i\}$ are in involution with respect to the Poisson tensor π then the functions $\{H_i \circ q,\ \tilde{H}_i = \sum_{s=1}^{N} \dfrac{\partial H_i}{\partial x_s}(\mathbf{x}) y_s\}$ are in involution with respect to the Poisson tensor $\pi_{TM,\lambda}$.

Toda lattice — bi-Hamiltonian system

The Hamiltonian

$$H = \sum_{i \in \mathbb{Z}} \left(\frac{1}{2} p_i^2 + e^{q_{i-1} - q_i} \right).$$

Hamilton's equations

$$\begin{cases} \dot{q}_i = \{q_i, H\} = p_i \\ \dot{p}_i = \{p_i, H\} = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}} \end{cases}.$$

Under Flaschka's transformation

$$a_i = \frac{1}{2} e^{\frac{(q_{i-1} - q_i)}{2}}, \quad b_i = -\frac{1}{2} p_{i-1}$$

the system transforms to

$$\begin{aligned} \frac{da_i}{dt} &= a_i \left(b_{i+1} - b_i \right), \\ \frac{db_i}{dt} &= 2 \left(a_i^2 - a_{i-1}^2 \right). \end{aligned}$$

The Toda lattice is equivalent to the Lax equation

$$\frac{dL}{dt} = [A, L],$$

where

$$Lf_i = a_i f_{i+1} + b_i f_i + a_{i-1} f_{i-1},$$

$$Af_i = a_i f_{i+1} - a_{i-1} f_{i-1}$$

are linear operators in the Hilbert space of square summable sequences $l^2(\mathbb{Z})$.

The Toda lattice is a bi-Hamiltonian system. There exist another Poisson bracket, which we denote by π_2, and another function H_1, which will play the role of the Hamiltonian for the π_2 bracket, such that $\pi_1 + \pi_2$ is Poisson tensor and $\pi_1 \nabla H = \pi_2 \nabla H_1$ $(H = \sum_i \left(2b_i^2 + 4a_i^2\right))$. The Poisson tensor π_1 is given by the relations

$$8\{a_i, b_i\}_1 = -a_i, \quad 8\{a_i, b_{i+1}\}_1 = a_i.$$

For the Toda lattice the π_2 bracket (which appeared in a paper of M. Adler) is quadratic in the variables b_i,a_i and it is given by the relations

$$\{a_i, a_{i+1}\}_2 = \frac{1}{2} a_i a_{i+1}, \quad \{a_i, b_i\}_2 = -a_i b_i,$$

$$\{a_i, b_{i+1}\}_2 = a_i b_{i+1}, \quad \{b_i, b_{i+1}\}_2 = 2a_i^2$$

and all other brackets are zero.

Example- Extended Toda Lattice

Functions $H_k = TrL^k$ are the functions in involutions with respect to the both brackets. The above functions for $k = 1, 2, 3$ have the expressions

$$H_1 = trL = \sum_{i\in\mathbb{Z}} b_i,\ H_2 = 2H = trL^2 = \sum_{i\in\mathbb{Z}} \left(b_i^2 + 2a_i^2\right), \qquad (7)$$

$$H_3 = trL^3 = \sum_{i\in\mathbb{Z}} \left(b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}\right).$$

Now deformed tangent Poisson structure $\pi_{TM,\lambda}$ in local coordinates $a_i, b_i, n_i, m_i,\ i \in \mathbb{Z}$, is given by the relation

$$\{a_i, m_i\}_{TM,\lambda} = -\frac{1}{4}a_i, \qquad \{a_i, m_{i+1}\}_{TM,\lambda} = \frac{1}{4}a_i, \qquad (8)$$

$$\{b_i, n_i\}_{TM,\lambda} = \frac{1}{4}a_i, \qquad \{b_{i+1}, n_i\}_{TM,\lambda} = -\frac{1}{4}a_i, \qquad (9)$$

$$\{n_i, n_{i+1}\}_{TM,\lambda} = \frac{\lambda}{2}a_i a_{i+1}, \qquad \{n_i, m_i\}_{TM,\lambda} = -\frac{1}{4}n_i - \lambda a_i b_i, \qquad (10)$$

From the last theorem we transform the functions $H_k = TrL^k$ into the functions $H_k \circ q_M^* = TrL^k \circ q_M^*$ and $\tilde{H}_k = \sum_{s=1}^{N} \left(\frac{\partial H_k}{\partial a_s} n_s + \frac{\partial H_k}{\partial b_s} m_s \right)$, i.e.

$$H_1 = \sum_{i \in \mathbb{Z}} b_i, \qquad \tilde{H}_1 = \sum_{i \in \mathbb{Z}} m_i,$$

$$H_2 = \sum_{i \in \mathbb{Z}} \left(b_i^2 + 2a_i^2\right), \qquad \tilde{H}_2 = \sum_{i \in \mathbb{Z}} \left(2b_i m_i + 4a_i n_i\right), \tag{11}$$

$$H_3 = \sum_{i \in \mathbb{Z}} \left(b_i^3 + 3a_i^2 b_i + 3a_i^2 b_{i+1}\right), \quad \tilde{H}_3 = \sum_{i \in \mathbb{Z}} \Big(3b_i^2 m_i + 3a_i^2 m_i + 3a_i^2 m_{i+}$$

$$+6a_i b_i n_i + 6a_i b_{i+1} n_i\Big),$$

$$\ldots \qquad \ldots$$

Now if we take as the Hamiltonian

$$H = \alpha H_2 + \beta \tilde{H}_2 = \sum_{i\in\mathbb{Z}} \left(\alpha b_i^2 + 2\alpha a_i^2 + 2\beta b_i m_i + 4\beta a_i n_i\right) \quad (12)$$

then Hamilton's equations are in the form

$$\begin{aligned}
\frac{da_i}{dt} &= \frac{1}{2}\beta a_i \left(b_{i+1} - b_i\right), \\
\frac{db_i}{dt} &= \beta \left(a_i^2 - a_{i-1}^2\right), \\
\frac{dn_i}{dt} &= \frac{1}{2}\alpha a_i \left(b_{i+1} - b_i\right) + \frac{1}{2}\beta a_i \left(m_{i+1} - m_i\right) + \frac{1}{2}\beta n_i \left(b_{i+1} - b_i\right) + \\
&\quad + 2\beta\lambda a_i \left(a_{i+1}^2 - a_{i-1}^2 - b_i^2 + b_{i+1}^2\right), \\
\frac{dm_i}{dt} &= \alpha \left(a_i^2 - a_{i-1}^2\right) + 2\beta \left(a_i n_i - a_{i-1} n_{i-1}\right) + \\
&\quad + 4\beta\lambda \left(a_i^2 b_{i+1} + a_i^2 b_i - a_{i-1}^2 b_i - a_{i-1}^2 b_{i-1}\right).
\end{aligned} \quad (13)$$

We can interpret these equations as an extension of the Toda lattice. It is the integrable system, where the integrals of motions are given by formulas (11). If we put $\alpha = \lambda = 0, \beta = 2$ and we take $n_i = m_i = 0$ then we observe that we reduce it to Toda lattice

Thank you for your attention