Trautman-Bondi energy and its universality

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1916 – Einstein describes gravitational waves. He uses linearized version of general relativity theory and derives his famous “quadrupole formula”.

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

Minkowski flat metric \[ g \]

20’ – 30’ – Einstein begins to have doubts about validity of such an approach:

\[ g = -dt^2 + dx^2 + dy^2 + dz^2 \\
- \cos(x - t) (2 + \cos(x - t)) \, dt^2 \\
+ 2 \cos(x - t) (1 + \cos(x - t)) \, dt \, dx - \cos^2(x - t) \, dx^2 \]

Minkowski flat metric perturbation

perturbation
Gravitational waves

\[ g = -dt^2 + dx^2 + dy^2 + dz^2 \]
\[ - \cos(x - t) (2 + \cos(x - t)) dt^2 \]
\[ + 2 \cos(x - t) (1 + \cos(x - t)) dtdx - \cos^2(x - t) dx^2 \]

Choose a new variable:

\[ \tilde{t} := t + \sin(t - x) \]

\[ g = -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 \]

Main difficulty: how to decouple "gauge freedom" from the "true degrees of freedom". (Einstein writes several papers.)

1937 – Einstein "proves" that gravitational waves do not exist!

"Red light": serious people do not believe in gravitational waves (including Leopold Infeld – father of the polish theoretical physics.)
Gravitational waves

``Red light’’: serious people do not believe in gravitational waves (including Leopold Infeld – father of the polish theoretical physics.)

1958–Andrzej Trautman Ph.D thesis: formulation of conceptual framework and mathematical tools which are necessary to describe gravitational radiation.

1) Radiation is not a local phenomenon.
2) It is localized ```at infinity”.
3) Boundary conditions are fundamental: generalization of the ```Sommerfeld radiation conditon” in special relativity (```asymptotic flatness” in General Relativity).

Compactified space-time

To each 2-dimensional slice we may assign energy which has not been radiated yet ("Trautman-Bondi energy").
Compactified space-time

Idea of the conformal compactification:

\[
\begin{align*}
g & = -dt^2 + dx^2 + dy^2 + dz^2 \\
& = -dt^2 + dr^2 + r^2 d\sigma^2 \\
u & := t - r, \quad v := t + r. \\
g & = -dudv + \frac{1}{4} (v - u)^2 d\sigma^2 \\
u & := \tan U, \quad v := \tan V, \quad -\pi/2 < U, V < \pi/2.
\end{align*}
\]

\[
g = \frac{1}{4 \cos^2 U \cos^2 V} \left[ -dUdV + \frac{1}{4} \sin^2 (V - U) d\sigma^2 \right]
\]

\[
T = U + V, \quad R = V - U.
\]

\[
g = \frac{1}{4 \cos^2 U \cos^2 V} \left[ -dT^2 + dR^2 + \frac{1}{4} \sin^2 R d\sigma^2 \right]
\]

\[
T \in [-\pi, \pi], \quad R \in [0, \pi - |t|].
\]
Compactified space-time

\[ g = \frac{1}{4 \cos^2 U \cos^2 V} \left[ -dT^2 + dR^2 + \frac{1}{4} \sin^2 R d\sigma^2 \right] \]

future timelike infinity \( T = +\pi, \quad R = 0 \)

future null infinity \( T + R = \pi \)

spacelike infinity \( T = 0, \quad R = \pi \)

past null infinity \( R - T = \pi \)

\( T \in [-\pi, \pi/2], \quad R \in [0, \pi - |t|] \).
Compactified space-time

$\mathcal{I}^+$ future timelike infinity \quad $T = +\pi$, \quad $R = 0$

$\mathcal{I}^+$ future null infinity \quad $T + R = \pi$

$I^0$ spacelike infinity \quad $T = 0$

$I^0$ past null infinity \quad $R = \pi$

$\mathcal{I}^-$ past timelike infinity \quad $T = -\pi$, \quad $R = 0$

$\mathcal{I}^-$ past null infinity \quad $R - T = \pi$

Conformal factor

$$g = \frac{1}{4 \cos^2 U \cos^2 V} \left[ -dT^2 + dR^2 + \frac{1}{4} \sin^2 R d\sigma^2 \right]$$
Compactified space-time

\[ \mathcal{I}^+ \text{ future timelike infinity} \quad T = +\pi, \quad R = 0 \]

\[ \mathcal{I}^+ \text{ future null infinity} \quad T + R = \pi \]

\[ \mathcal{I}^0 \text{ spacelike infinity} \quad T = 0 \quad R = \pi \]

\[ \mathcal{I}^- \text{ past null infinity} \quad R - T = \pi \]

\[ \mathcal{I}^- \text{ past timelike infinity} \quad T = -\pi, \quad R = 0 \]

conformal factor

\[ g = \Omega^{-2} \left[ -dT^2 + dR^2 + \frac{1}{4} \sin^2 R d\sigma^2 \right] \]
Compactified space-time

I$^+$ future timelike infinity  \( T = +\pi \),  \( R = 0 \)

\( T + R = \pi \)

\( T = 0 \)

\( R = \pi \)

I$^-$ past timelike infinity  \( T = -\pi \),  \( R = 0 \)

\( R - T = \pi \)

I$^0$ spacelike infinity

I$^0$ future null infinity

I$^-$ past null infinity

conformal factor

\[ g = \Omega^{-2} \tilde{g} \]

fictitious metric

June 2, 2018  Geometry, Integrability and Quantization
Compactified space-time

\[ g = \Omega^{-2} \tilde{g} \]

fictitious metric

Outgoing radiation

Incoming radiation
Universal properties of radiation

Asymptotic behaviour of radiation is universal - not specific to general relativity.

In asymptotic regime: dynamic of gravitational field = two decoupled wave equations.

Penrose' metric description (fictitious!) of the "Scri" is obsolete.

Consider wave equation on Minkowski space:

\[ \square \phi = \left( -\frac{d^2}{dt^2} + \Delta \right) \phi = 0 \]
Wave equation

\[ \Box \phi = \left( -\frac{d^2}{dt^2} + \Delta \right) \phi = 0 \]

Every solution of wave equation is uniquely determined by initial (Cauchy) data:

\[ \phi(0, \vec{x}) = \varphi(\vec{x}) , \quad \frac{\partial}{\partial t} \phi(0, \vec{x}) = \pi(\vec{x}) . \]

\[ \phi(t, \vec{x}) := \frac{\partial}{\partial t} \left( t \cdot \overline{\varphi(S(\vec{x}, |t|))} \right) + t \cdot \overline{\pi(S(\vec{x}, |t|))} . \]

Mean value of the function over the sphere centered at \( \vec{x} \) whose radius is: \( |t| \)

„Huygens formula“
Wave equation

\[ \phi(t, \bar{x}) := \frac{\partial}{\partial t} \left( t \cdot \overline{\varphi(S(\bar{x}, |t|))} \right) + t \cdot \overline{\pi(S(\bar{x}, |t|))} . \]
Point on the scri – “endpoint” of a light ray.
Finite energy!

\[ F(s) = \lim_{r \to \infty} r \cdot \phi(r, 0, 0, r) \]

Point on the scri – “endpoint” of a light ray.
Wave equation

\[ F(s) = \lim_{r \to \infty} r \cdot \phi(r, 0, 0, r) \]

Finite energy!
Radiation data on Scri

Null hyperplane

\(\Box \phi = -4 \left( \frac{\partial^2 \phi}{\partial u \partial v} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \equiv 0\)

\(\frac{\partial}{\partial v} \int \frac{\partial \phi}{\partial u} \, dx \, dy = 0\)

\(F(s) = \int \frac{\partial \phi}{\partial u} \, dx \, dy\)
Radiation data on Scri

Null hyperplane

\[ u = t - z \]

\[ v = t + z \]

\[ \frac{\partial}{\partial u} \int \frac{\partial \phi}{\partial v} dxdy = 0 \]

\[ F(s) = \int \frac{\partial \phi}{\partial v} dxdy \]
To parameterize the set of all null planes (points of „scri”): choose a time axis.

1) Intersection of „s” with a chosen time axis.
2) Two angles describing all the light rays starting from a point („celestial sphere”)

Geometry, Integrability and Quantization
Coordinate system on the scri: \((\tau, \theta, \varphi) \in \mathbb{R}^1 \times S^2\)
**Theorem:** Transformation between initial (Cauchy) data \((\pi, \varphi)\) and the radiation data \(F\) is a symplectomorphism (canonical transformation) of the two canonical structures:

\[
\Omega_{\text{Cauchy}} = \int \delta \pi(\vec{x}) \wedge \delta \varphi(\vec{x}) d^3x
\]

\[
\Omega_{\text{Cauchy}}((\pi, \varphi), (\pi', \varphi')) = \int (\pi(\vec{x})\varphi'(\vec{x}) - \pi'(\vec{x})\varphi(\vec{x})) d^3x
\]

\[
\Omega_{\text{Radiation}} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F d\tau d\mu
\]

\[
\Omega_{\text{Radiation}}(F, G) = \int \frac{\partial F}{\partial \tau} G d\tau d\mu
\]

\[
d\mu = \sin \theta d\theta d\varphi
\]

The two structures can be treated as different representations of the same phase space describing possible field configurations.
Initial data vs. radiation data

**Theorem:** Transformation between initial (Cauchy) data \((\pi, \varphi)\) and the radiation data \(F\) is a symplectomorphism (canonical transformation) of the two canonical structures:

\[
\Omega_{\text{Cauchy}} = \int \delta \pi(\vec{x}) \wedge \delta \varphi(\vec{x}) d^3x
\]

\[
\{\pi(\vec{x}), \varphi(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})
\]

\[
\Omega = \delta p \wedge \delta q
\]

\[
\Omega_{\text{Radiation}} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F d\tau d\mu
\]

\[
\{F(\tau, \theta, \varphi), F(\tilde{\tau}, \tilde{\theta}, \tilde{\varphi})\} = \delta(\theta - \tilde{\theta})\delta(\varphi - \tilde{\varphi})\delta'(\tau - \tilde{\tau})
\]

\[
d\mu = \sin \theta d\theta d\varphi
\]

The two structures can be treated as different representations of the same phase space describing possible field configurations.
Lorentz invariance

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F \, d\tau \, d\mu \]

Symplectic structure in space of radiation data is not, a priori, Lorentz invariant: depends upon the choice of the time axis!

What happens if we change the time axis?

1) Time parameter changes.

2) Change of the volume element \( d\mu \) on the celestial sphere.

3) Change of the value of 

\[ F(s) = \lim_{r \to \infty} r \cdot \phi(r, 0, 0, r) \]

But, the miracle occurs:

\[ \mathcal{F} := F \cdot \sqrt{d\mu} \]

remains unchaned!!!
Field energy

Time evolution of the field is generated by the Hamiltonian (field energy).

In "Cauchy picture":
\[ \dot{\varphi} = \pi = \frac{\delta \mathcal{H}}{\delta \pi} \]
\[ \dot{\pi} = \Delta \varphi = -\frac{\delta \mathcal{H}}{\delta \varphi} \]

\[ \Omega_{Cauchy} = \int \delta \pi \wedge \delta \varphi \]
\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) d^3x \geq 0 \]
\[ -\varphi \Delta \varphi \]

In "radiation picture":

\[ \dot{\mathcal{F}} = \partial_\tau \mathcal{F} = \frac{\delta \mathcal{H}}{\delta \mathcal{P}} \]
\[ \dot{\mathcal{P}} = \partial_\tau \mathcal{P} = -\frac{\delta \mathcal{H}}{\delta \mathcal{F}} \]

\[ \Omega_{Radiation} = \int \delta \left( \frac{\partial \mathcal{F}}{\partial \tau} \right) \wedge \delta \mathcal{F} d\tau \]

\[ \mathcal{H} = \int (\mathcal{P} \partial_\tau \mathcal{F}) d\tau = \int (\partial_\tau \mathcal{F})^2 d\tau \geq 0 \]

(like momentum in the conventional Cauchy formulation)
Time evolution

\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) d^3x \]
\( \mathcal{H} = \int (\partial_\tau \mathcal{F})^2 d\tau \)

\( \mathcal{H} = \frac{1}{2} \int (\pi^2 + (\nabla \varphi)^2) \, d^3 x \)
Time evolution

\[ \mathcal{H} = \int (\partial_\tau \mathcal{F})^2 d\tau \]

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial \mathcal{F}}{\partial \tau} \wedge \delta \mathcal{F} d\tau \]

\[ \Omega_{\text{Cauchy}} = \int \delta \pi \wedge \delta \varphi \]

\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) d^3x \]

Quantization???
Cauchy problem on a hyperboloid

\[ \mathcal{H} = \int (\partial_\tau F)^2 \, d\tau \]

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F \, d\tau \]

\[ \Omega_{\text{Cauchy}} = \int \delta \pi \wedge \delta \varphi \]

\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) \, d^3 x \]
Cauchy problem on a hyperboloid

\[ \mathcal{H} = \int (\partial_\tau \mathcal{F})^2 d\tau \]

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial \mathcal{F}}{\partial \tau} \wedge \delta \mathcal{F} d\tau \]

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\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) d^3x \]
Mixed: „Cauchy–Radiation” picture

\[ \mathcal{H} = \int (\partial_\tau \mathcal{F})^2 \, d\tau \]

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\[ \mathcal{H} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) \, d^3x \]
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\[ \mathcal{H} = \int (\partial_\tau \mathcal{F})^2 d\tau \]

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial \mathcal{F}}{\partial \tau} \wedge \delta \mathcal{F} d\tau \]

\[ \Omega_{\text{Cauchy}} = \int \delta \pi \wedge \delta \varphi \]

\[ \mathcal{H} = \frac{1}{2} \int (\pi^2 + (\nabla \varphi)^2) d^3x \]
Mixed: „Cauchy–Radiation” picture

\[ \mathcal{F} \]

\((\varphi, \pi)\)

\[ \mathcal{H}^- = \int (\partial_\tau \mathcal{F})^2 \, d\tau \]

\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial \mathcal{F}}{\partial \tau} \wedge \delta \mathcal{F} \, d\tau \]

\[ \Omega_{\text{Cauchy}} = \int \delta \pi \wedge \delta \varphi \]

\[ \mathcal{H}^+ = \frac{1}{2} \int (\pi^2 + (\nabla \varphi)^2) \, d^3 x \]

\[ \mathcal{H}^{\text{total}} = \mathcal{H}^- + \mathcal{H}^+ \]
Mixed: „Cauchy–Radiation” picture

\[ \mathcal{H}^{-} = \int (\partial_{\tau} F)^2 \, d\tau \]
\[ \Omega_{\text{Radiation}} = \int \delta \frac{\partial F}{\partial \tau} \wedge \delta F \, d\tau \]

\[ \Omega_{\text{Cauchy}} = \int \delta \pi \wedge \delta \varphi \]
\[ \mathcal{H}^{+} = \frac{1}{2} \int \left( \pi^2 + (\nabla \varphi)^2 \right) \, d^3 x \]

\[ \mathcal{H}^{total} = \mathcal{H}^{-} + \mathcal{H}^{+} \]
\[ \frac{d}{dt} \mathcal{H}^{total} = 0 \]
\[ \mathcal{H}^{-}(t) = \int_{-\infty}^{t} (\partial_{\tau} F)^2 \, d\tau \]

\( \mathcal{H}^{-} \) is strictly increasing, whence, \( \mathcal{H}^{+} \) is strictly decreasing
$H^{total} = H^- + H^+ = \text{const.}$

$H^+ \rightarrow 0$  Trautman-Bondi energy

$H^- = \int (\partial_\tau F)^2 \, d\tau$

$H^-(t) = \int_{\tau<t} (\partial_\tau F)^2 \, d\tau$

$\frac{d}{dt} H^+(t) = -\frac{d}{dt} H^-(t) = - (\partial_t F)^2$
Technical features which are specific to General Relativity:

No null hyper-planes (but asymptotically…)

Meaningful physical quantities never defined by volume integrals but only by surface integrals.

Hence: shape of the hyperbolid does not matter!

What matters is its intersection with the Scri.

Cauchy energy density is given by a complete divergence of a „superpotential” (i.e. „Freud”, „Landau-Lifshitz” or any other) and, whence, Trautman-Bondi energy can be calculated as a surface integral over any 2D surface on Scri.
Mixed: „Cauchy–Radiation” picture
Mixed: „Cauchy–Radiation” picture

\[ \mathcal{H}^{total} = \mathcal{H}^- + \mathcal{H}^+ = \text{const.} \]

Trautman-Bondi energy

already radiated energy