The Heisenberg group and $SL_2(\mathbb{R})$

a survival pack

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$SL_2(\mathbb{R})$ and Its Subgroups

$SL_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and $\det = 1$. A two dimensional subgroup F (F') of lower (upper) triangular matrices:

$$F = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \right\}, \qquad F' = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}, \qquad a \in \mathbb{R}_+, \; b, c \in \mathbb{R}.$$

F is isomorphic to the group of affine transformations of the real line ($ax + b$ group), isomorphic to the upper half-plane.

There are also three one dimensional continuous subgroups:

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \; t \in \mathbb{R} \right\}, \tag{1}$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \; t \in \mathbb{R} \right\}, \tag{2}$$

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \; t \in (-\pi, \pi] \right\}. \tag{3}$$

… and Nothing Else
(up to a conjugacy)

Proposition 1.
Any one-parameter continuous subgroup of $SL_2(\mathbb{R})$ *is conjugate to either* A, N *or* K.

Proof.
Any one-parameter subgroup is obtained through the exponentiation

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \tag{4}$$

of an element X of the Lie algebra $\mathfrak{sl}_2$ of $SL_2(\mathbb{R})$. Such X is a 2×2 matrix with the zero trace. The behaviour of the Taylor expansion (4) depends from properties of powers X^n. This can be classified by a straightforward calculation.

Lemma 2.
The square X^2 *of a traceless matrix* $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *is the identity matrix times* $a^2 + bc = -\det X$*. The factor can be negative, zero or positive, which corresponds to the three different types of the Taylor expansion* (4) *of* $e^{tX} = \sum \frac{t^n}{n!} X^n$.

It is a simple exercise in the Gauss elimination to see that through the matrix similarity we can obtain from X a generator

- of the subgroup K if $(-\det X) < 0$;
- of the subgroup N if $(-\det X) = 0$;
- of the subgroup A if $(-\det X) > 0$.

The determinant is invariant under the similarity, thus these cases are distinct.

$\mathrm{SL}_2(\mathbb{R})$ and Homogeneous Spaces

Let G be a group and H be its closed subgroup.
The *homogeneous space* G/H from the equivalence relation: $g' \sim g$ iff $g' = gh$, $h \in \mathsf{H}$. The *natural projection* $p : \mathsf{G} \rightarrow \mathsf{G}/\mathsf{H}$ puts $g \in \mathsf{G}$ into its equivalence class.
A continuous section $s : \mathsf{G}/\mathsf{H} \rightarrow \mathsf{G}$ is a right inverse of p, i.e. $p \circ s$ is an identity map on G/H. Then the *left action* of G on itself:

$$\Lambda(g) : g' \mapsto g^{-1} * g', \quad \text{generates} \quad \begin{array}{ccc} G & \xrightarrow{g*} & G \\ s\uparrow\downarrow p & & s\uparrow\downarrow p \\ G/H & \xrightarrow{g\cdot} & G/H \end{array}$$

If $\mathsf{G} = \mathrm{SL}_2(\mathbb{R})$ and $\mathsf{H} = \mathsf{F}$, then $\mathrm{SL}_2(\mathbb{R})/\mathsf{F} \sim \mathbb{R}$ and $p : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{b}{d}$:

$$s : u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad g : u \mapsto p(g^{-1} * s(u)) = \frac{au+b}{cu+d}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$SL_2(\mathbb{R})$ and Imaginary Units

Consider $G = SL_2(\mathbb{R})$ and H be any of 1D subgroups A, N or K. A right inverse s to the natural projection $p : G \to G/H$:

$$s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2, \text{in the diagram} \quad \begin{array}{ccc} G & \xrightarrow{g*} & G \\ s\uparrow\downarrow p & & s\uparrow\downarrow p \\ G/H & \xrightarrow{g\cdot} & G/H \end{array}$$

defines the G-action $g \cdot x = p(g \cdot s(x))$ on the homogeneous space G/H:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \left(\frac{(au + b)(cu + d) - \sigma cav^2}{(cu + d)^2 - \sigma(cv)^2}, \frac{v}{(cu + d)^2 - \sigma(cv)^2} \right).$$

This becomes a Möbius map in (hyper)complex numbers:[1]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \qquad w = u + iv, \quad i^2 (:= \sigma) = -1, 0, 1.$$

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[1] Kisil, *Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of* $SL_2(\mathbf{R})$, 2012.

Structural Equivalence Principle

During this course we will see many illustrations to the following:
Structural Equivalence Principle—SEP:
The structure of the group $SL_2(\mathbb{R})$ and its representations are interchangeable by simultaneous choice of one-dimensional subgroup K, N' or A' and the corresponding hypercomplex unit i, ε or j, see Table 1.

Case:	elliptic	parabolic	hyperbolic
Numbers	complex	dual	double
Subgroup H	K	N	A
$\sigma = \iota^2$	-1	0	1

Table: Correspondence between components of the construction

Möbius Transformations of $\mathbb{R}^2$

For *all* numbers define *Möbius' transformation* of $\mathbb{R}^2 \to \mathbb{R}^2$, (in elliptic and parabolic cases this is even $\mathbb{R}^2_+ \to \mathbb{R}^2_+$!):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ u + iv \ \mapsto \ \frac{a(u+iv)+b}{c(u+iv)+d}. \tag{5}$$

Product $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ gives *Iwasawa* $SL_2(\mathbb{R}) = ANK$. In all $\mathbb{A}$ subgroups A and N acts uniformly:

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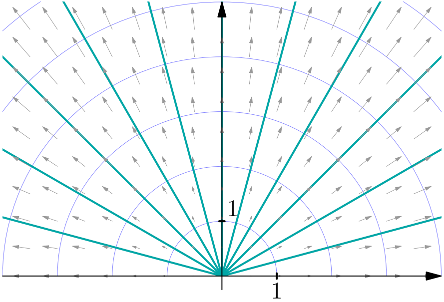

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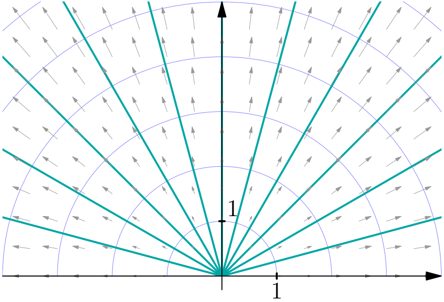
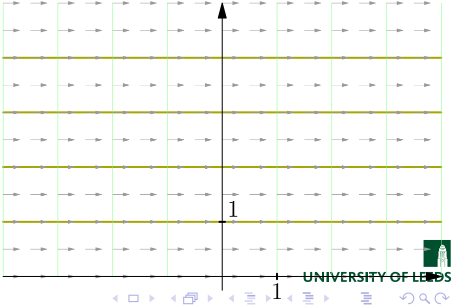
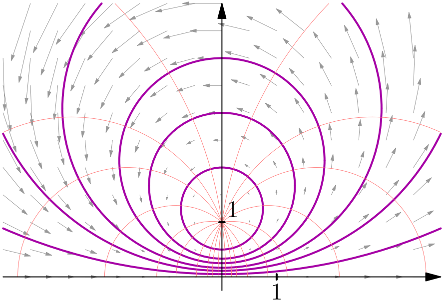


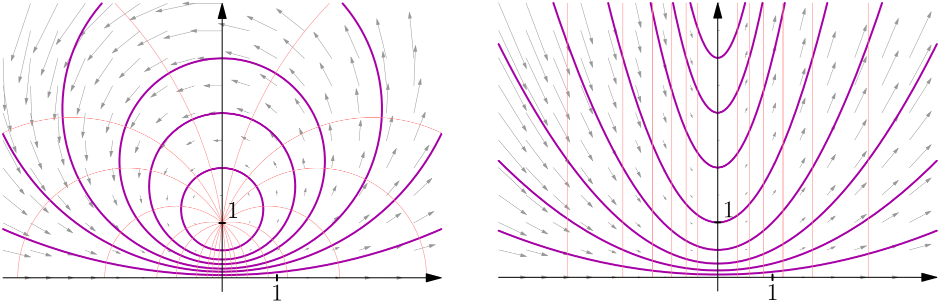




Vector fields are:

$$
\begin{aligned}
dK_e(u,v) &= (1+u^2-v^2, \quad 2uv)\\
dK_p(u,v) &= (1+u^2, \quad 2uv)\\
dK_h(u,v) &= (1+u^2+v^2, \quad 2uv)\\
dK_\sigma(u,v) &= (1+u^2+\sigma v^2, \quad 2uv)
\end{aligned}
$$

Figure: Depending from $i^2=\sigma$ the orbits of subgroup K are circles, parabolas and hyperbolas passing $(0,t)$ with the equation $(u^2-\sigma v^2)+v(\sigma t-t^{-1})+1=0$. This leads to elliptic, parabolic and hyperbolic analytic functions.



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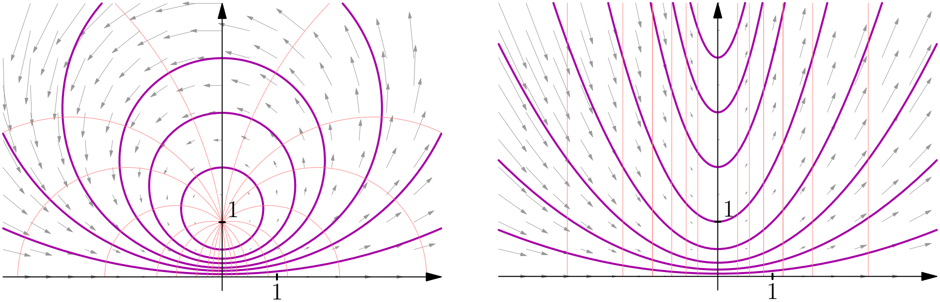

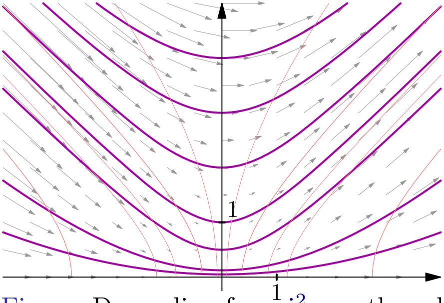

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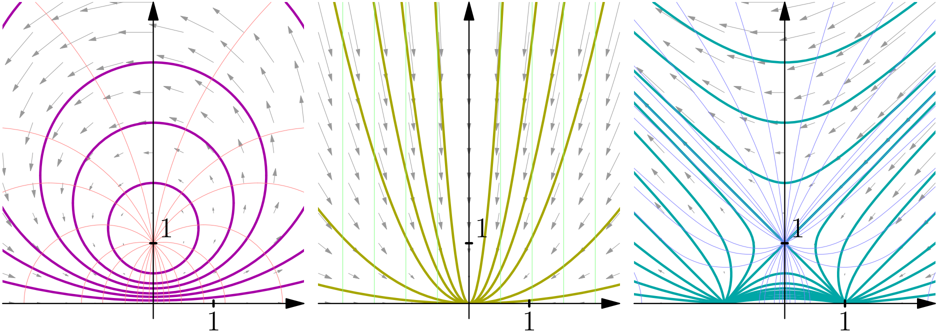
Fix subgroups of i, ε and j



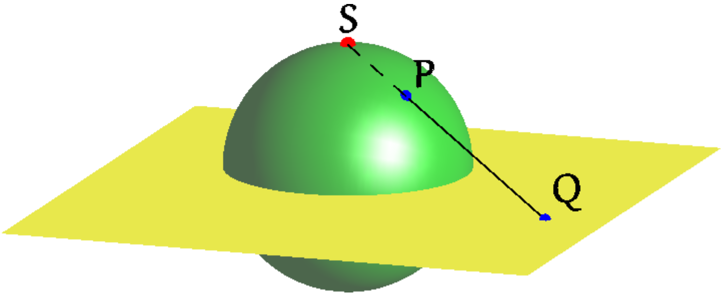
$$K = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$$

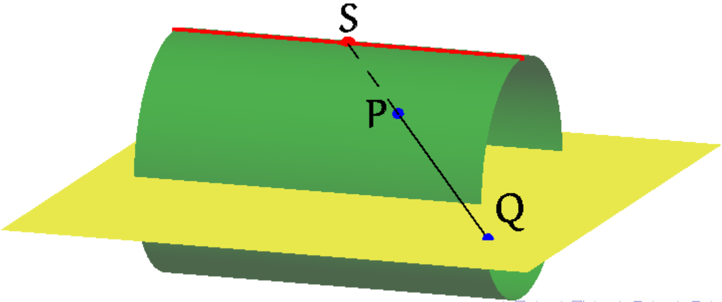
$$N' = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

Fix subgroups of $\iota = (0, 1)$ are $S(t) = \exp \begin{pmatrix} 0 & \sigma t \\ t & 0 \end{pmatrix}$, where $\sigma = \iota^2$.

Compactification of $\mathbb{R}^e$ and $\mathbb{R}^p$





Compactification of $\mathbb{R}^h$

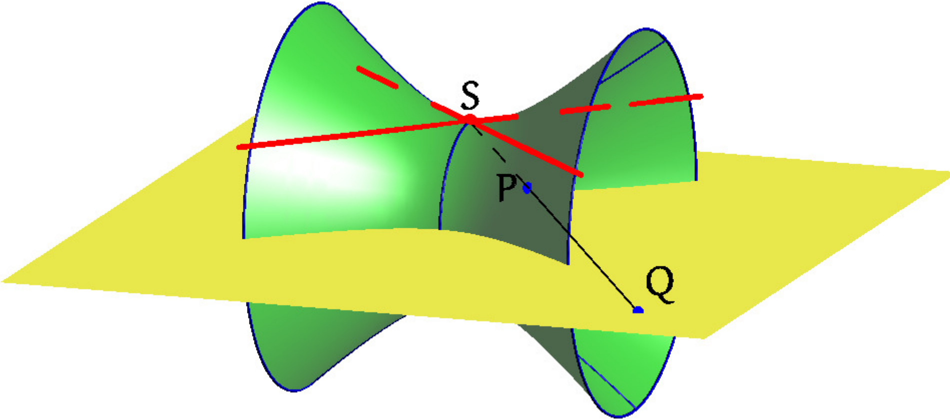


Figure: Hyperbolic counterpart of the Riemann sphere (incomplete so far!) Ideal elements for the *light cone* at infinity.

In all EPH cases ideal points comprise the corresponding zero-radius cycle at infinity.

Induced Representations

Let G be a group, H its closed subgroup, χ be a linear representation of H in a space V. The set of V-valued functions with the property

$$F(gh) = \chi(h)F(g),$$

is invariant under left shifts.
The restriction of the left regular representation to this space is called an *induced representation*.
Equivalently we consider the *lifting* of $f(x)$, $x \in X = G/H$ to $F(g)$:

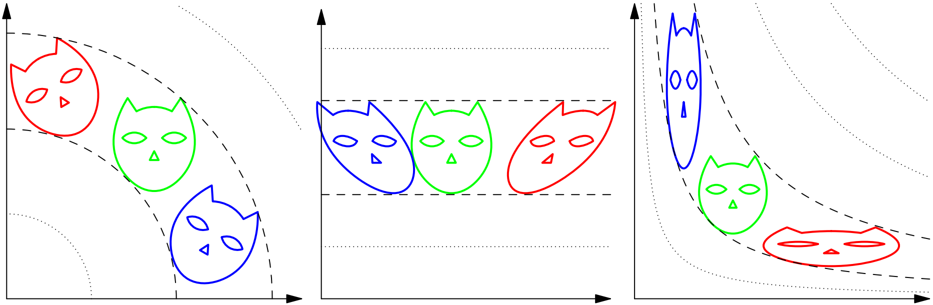
$$F(g) = \chi(h)f(p(g)), \qquad p : G \to X, \quad g = s(x)h, \quad p(s(x)) = x.$$

This is a 1-1 map which transform the left regular representation on G to the following action:

$$[\rho'(g)f](x) = \chi(h)f(g \cdot x), \qquad \text{where} \quad gs(x) = s(g \cdot x)h.$$

In the case of $SL_2(\mathbb{R})$ we have three different types of actions.

Characters and transformations of $\mathbb{R}^2$



Multiplication by an unimodular complex number is an orthogonal rotation of $\mathbb{R}^2$. Multiplication by unimodular dual and double numbers can be viewed as parabolic and hyperbolic rotations[2] preserving the area (i.e. the symplectic form). They induce some representations as well.

[2]Yaglom, *A Simple Non-Euclidean Geometry and Its Physical Basis*, 1979.

Affine Group

For $G = SL_2(\mathbb{R})$ and $H = F$ the action on G/H is:

$$g : u \mapsto p(g^{-1} * s(u)) = \frac{au+b}{cu+d}, \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We calculate also that

$$r(g^{-1} * s(u)) = \begin{pmatrix} (cu+d)^{-1} & 0 \\ c & cu+d \end{pmatrix}.$$

A generic character of F is a power of its diagonal element:

$$\rho_\kappa \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^\kappa.$$

Thus the corresponding realisation of induced representation is:

$$\rho_\kappa(g) : f(u) \mapsto \frac{1}{(cu+d)^\kappa} f\left(\frac{au+b}{cu+d}\right) \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (6)$$

Induced Wavelet Transform

Let $v_0 \in \mathcal{H}$ be an eigenfunction as follows:

$$\rho(h)v_0 = \tilde{\chi}(h) \cdot v_0, \qquad \text{for all } h \in \tilde{H}.$$

It is suitable to be the *mother wavelet* (*vacuum vector*). Then we have

$$\begin{aligned}
[\mathcal{W}f](gh) &= \langle f, \rho(gh)v_0\rangle = \langle f, \rho(g)\rho(h)v_0\rangle \\
&= \langle f, \tilde{\chi}(h) \cdot \rho(g)v_0\rangle = \tilde{\chi}(h^{-1}) \langle f, \rho(g)v_0\rangle .
\end{aligned}$$

For v_0 the *induced wavelet transform* $\mathcal{W} : \mathcal{H} \to L_\infty(G/\tilde{H})$ by

$$[\mathcal{W}f](w) = \langle f, \rho_0(s(w))v_0\rangle , \tag{7}$$

where $w \in G/\tilde{H}$ and $s : G/\tilde{H} \to G$.
It intertwines ρ with a representation induced by $\tilde{\chi}^{-1}$ of $\tilde{H}$.
Particularly, it intertwines ρ with the representation associated to G-action on the homogeneous space $G/\tilde{H}$.

The Lie algebra $\mathfrak{sl}_2$ of $\mathrm{SL}_2(\mathbb{R})$ consists of all 2×2 real matrices of trace zero. One can introduce a basis:

$$A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

The commutator relations are

$$[Z, A] = 2B, \qquad [Z, B] = -2A, \qquad [A, B] = -\frac{1}{2}Z.$$

The derived representation for a vector field $Y \in \mathfrak{sl}_2$ is defined through the exponential map $\exp : \mathfrak{sl}_2 \to \mathrm{SL}_2(\mathbb{R})$ by the standard formula:

$$d\rho^Y = \frac{d}{dt}\rho(e^{tY})\Big|_{t=0}. \tag{9}$$

Example 1 (the derived representation of (6)).

For $A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we get $(e^{tA})^{-1} = e^{-tA} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. Thus:

$$d\rho^A f(u) = \frac{d}{dt}[\rho(e^{tA})f](u)\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{e^{-\kappa t/2}} f\left(e^t u\right)\right]\Big|_{t=0}$$
$$= \frac{\kappa}{2} f(u) + u f'(u).$$

Similarly, for the basis (8) of $\mathfrak{sl}_2$ the derived representation of (6) is:

$$d\rho^A_\kappa = \frac{\kappa}{2} \cdot I + u \cdot \partial_u, \qquad (10)$$

$$d\rho^B_\kappa = \frac{\kappa}{2} u \cdot I + \frac{1}{2}(u^2 - 1) \cdot \partial_u, \qquad (11)$$

$$d\rho^Z_\kappa = -\kappa u \cdot I - (u^2 + 1) \cdot \partial_u \qquad (12)$$

Cauchy–Riemann Equation

from Invariant Fields

Let ρ be a unitary representation of Lie group G with the derived representation $d\rho$ of $\mathfrak{g}$. Let a mother wavelet w_0 be a null-solution, i.e. $Aw_0 = 0$, for the operator $A = \sum_J a_j d\rho^{X_j}$, where $X_j \in \mathfrak{g}$. Then the wavelet transform $F(g) = \mathcal{W}f(g) = \langle f, \rho(g)w_0 \rangle$ for any f satisfies to:

$$DF(g) = 0, \qquad \text{where} \quad D = \sum_j a_j \mathfrak{L}^{X_j}.$$

Here $\mathfrak{L}^{X_j}$ are left the invariant fields (Lie derivatives) on G corresponding to X_j.

If $\mathfrak{L}^{X_j}$ is derived representation of Lie derivative A, N, K (without the matching subgroup) then C-R operator and Laplacian are given by:

$$D = \iota \mathfrak{L}^{A} + \mathfrak{L}^{X}, \qquad \text{and} \qquad \Delta = D\bar{D} = -\sigma {\mathfrak{L}^{A}}^2 + {\mathfrak{L}^{X}}^2, \qquad (13)$$

where X is in the orthogonal complement (with respect to the Killing form) of the corresponding subgroup K, N, A.

Consider the representation ρ

$$\rho_2(g) : f(u) \mapsto \frac{1}{(cu+d)^2} f\left(\frac{au+b}{cu+d}\right) \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let A and $N \in \mathfrak{sl}_2$ generates $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ and $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then the derived representations are:

$$[d\rho^A f](x) = f(x) + xf'(x), \qquad [d\rho^N f](x) = f'(x).$$

The corresponding left invariant vector fields on upper half-plane are:

$$\mathfrak{L}^A = a\partial_a, \qquad \mathfrak{L}^N = a\partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator $d\rho^A + i d\rho^N = I + (x+i)\frac{d}{dx}$. Therefore the wavelet transform will consist of the null solutions to the operator $\mathfrak{L}^A - i\mathfrak{L}^N = a(\partial_a + i\partial_b)$—the Cauchy-Riemann operator.

The infinitesimal version of the eigenvector property $\rho(h)v_0 = \chi(h) \cdot v_0$ is $d\rho_n^Z v_0 = \lambda v_0$, explicitly, cf. (12)

$$nuf(u) + f'(u)(1+u^2) = \lambda f(u).$$

The generic solution is:

$$f(u) = \frac{1}{(1+u^2)^{n/2}} \left(\frac{u+i}{u-i}\right)^{i\lambda/2} = \frac{(u+i)^{(i\lambda-n)/2}}{(u-i)^{(i\lambda+n)/2}}.$$

To avoid multivalent function we need to put $\lambda = im$ with an integer m. The Cauchy–Riemann condition (which turn to be later the same as "the minimal weight condition") suggests $m = n$. Thus, the induced wavelet transform is:

$$\hat{f}(x,y) = \langle f, \rho_n f_0 \rangle = \int_{\mathbb{R}} f(u) \frac{\sqrt{y}}{u - x - iy}\, dx = \sqrt{y} \int_{\mathbb{R}} f(u) \frac{dx}{u - (x+iy)}$$

And its image consists of null solutions of Cauchy–Riemann type equations. For $m > n$ we obtain *polyanalytic* functions annihilated by powers of Cauchy–Riemann operator.

Fix Subgroups of i and j

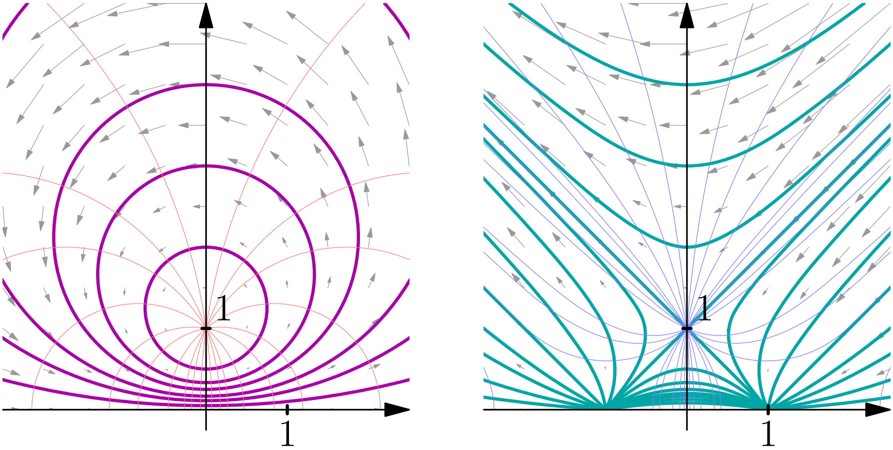


Figure: Elliptic and hyperbolic fix groups of the imaginary units.
In the hyperbolic case there are fixed geometric sets: $\{-1, 1\}$, $(-1, 1)$, $\mathbb{R}$.

For the subgroup A' generated by $B \in \mathfrak{sl}_2$ the derived representation, cf. (11):

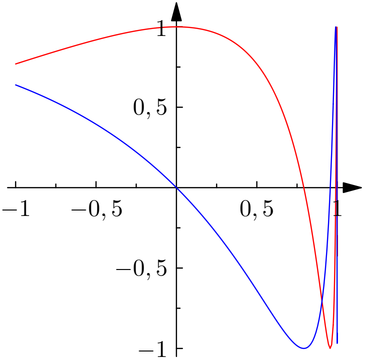
$$d\rho_n^B f(u) = -nuf(u) + (u^2 - 1)f'(u).$$

It has two singular point ± 1, its solution has compact support $[-1, 1]$.

$$\begin{aligned} f(x) &= \frac{1}{(u^2-1)^{n/2}} \left(\frac{u+1}{u-1}\right)^{\lambda/2} \\ &= \frac{(u+1)^{(\lambda-n)/2}}{(u-1)^{(\lambda+n)/2}}. \end{aligned}$$

For $\lambda = jm$ we also get, cf. K-case:

$$f(x) = \frac{(x+j)^{(m-\kappa)/2}}{(x-j)^{(m+\kappa)/2}}.$$



Hyperbolic Wavelets from Double Numbers

The choice of the A-eigenvector as mother wavelet:

- $f_0 = \delta(x \pm 1)$—Dirichlet condition.
- $f_0 = \frac{1}{(x-j)^\sigma} = \left(\frac{x+j}{x^2-1}\right)^\sigma$—Neumann condition.
- $f_0 = \frac{\chi(1-x^2)}{(x-j)^\sigma}$—space-like and time-like separation, Fig. 4.
- … (combination of above)

Then we follow the general scheme both for wavelets with complex and double valued wavelets:

- *wavelets* or *coherent states* $v_\sigma(g, z) = \rho_\sigma(g) v_0(z)$.
- d'Alembert integral from the universal *wavelet transforms*

$$\mathcal{W}_\sigma : f(z) \mapsto \mathcal{W}_\sigma f(u) = \langle f(z), \rho_\sigma v_0(u, z) \rangle$$

The subgroup N consists of shifts, the eigenfunction is $e^{\lambda u}$ and the induced wavelet transform coincides with the Fourier transform.
For the subgroup N', the generator is $d\rho_n^{Z/2-B} = (un)\cdot I - u^2 \cdot \partial_u$, cf. (11–12). The eigenvector $d\rho_n^{Z/2-B} f = \lambda f$ is $f_0(u) = u^n e^{\frac{\lambda}{u}}$.
Consider some identities for dual numbers:

$$e^{\varepsilon\alpha t} = 1 + \varepsilon\alpha t; \qquad (t \pm \varepsilon)^\alpha = t^{\alpha-1}(t \pm \varepsilon\alpha); \qquad (t-\varepsilon)(t+\varepsilon) = t^2.$$

Combining them together we can write for $\lambda = \varepsilon m$:

$$e^{\frac{\varepsilon m}{u}} = 1 + \frac{\varepsilon m}{u} = \left(\frac{u+\varepsilon}{u-\varepsilon}\right)^{m/2}$$

Then the solution $f_0(u) = u^n e^{\frac{\lambda}{u}}$ is:

$$|u|^{-\kappa} e^{-\frac{\varepsilon m}{u}} = \frac{1}{((u+\varepsilon)(u-\varepsilon))^{\kappa/2}} \left(\frac{u+\varepsilon}{u-\varepsilon}\right)^{m/2} = \frac{(u+\varepsilon)^{(m-\kappa)/2}}{(u-\varepsilon)^{(m+\kappa)/2}} \quad (14)$$

The respective wavelet transform is again very similar to the complex case.

Raising/Lowering Operators

Denote $\tilde{X} = d\rho(X)$ for $X \in \mathfrak{sl}_2$. Let $X = Z$ be the generator of the compact subgroup K, eigenspaces $\tilde{Z}v_k = ikv_k$ are parametrised by an integer $k \in \mathbb{Z}$. The raising/lowering operators $L_\pm$:

$$[\tilde{Z}, L_\pm] = \lambda_\pm L_\pm. \tag{15}$$

[$L_\pm$ are eigenvectors for operators $\mathrm{ad}\, Z$ of adjoint representation of $\mathfrak{sl}_2$.]
From the commutators (15) L_+v_k are eigenvectors of $\tilde{Z}$ as well:

$$\begin{aligned} \tilde{Z}(L_+v_k) &= (L_+\tilde{Z} + \lambda_+L_+)v_k = L_+(\tilde{Z}v_k) + \lambda_+L_+v_k \\ &= ikL_+v_k + \lambda_+L_+v_k = (ik + \lambda_+)L_+v_k. \end{aligned}$$

Thus those operators acts on a chain of eigenspaces:

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik-\lambda} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik+\lambda} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

Finding Raising/Lowering Operators

Elliptic and hyperbolic

Subgroup K. Assuming $L_+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ we obtain a linear equation:

$$c = 0, \qquad 2a = \lambda_+ b, \qquad -2b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 4 = 0$, and the raising operator is $L_+ = i\tilde{A} + \tilde{B}$.

Subgroup A. For the commutator $[\tilde{B}, L_+] = \lambda L_+$ we will got the system:

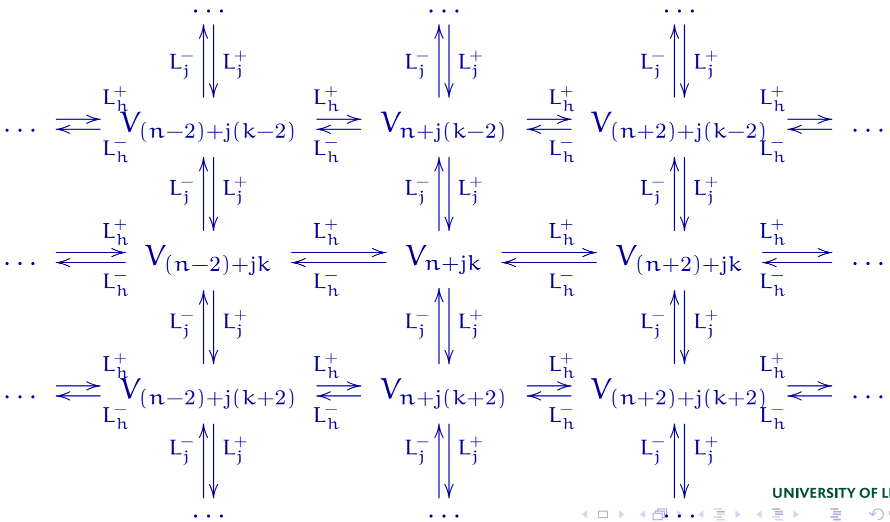
$$2c = \lambda a, \qquad b = 0, \qquad \frac{a}{2} = \lambda c.$$

A solution exists if and only if $\lambda^2 = 1$. The obvious values $\lambda = \pm 1$ with the operator $L_\pm = \pm\tilde{A} + \tilde{Z}/2$. Each indecomposable $\mathfrak{sl}_2$-module is formed by one-dimensional chain of eigenvalue with transitive action of raising/lowering operators.

Hyperbolic Ladder Operators

Double numbers: $\lambda = \pm j$ solves $\lambda^2 = 1$ additionally to $\lambda = \pm 1$.

The raising/lowering operators $L^h_\pm = \pm j\tilde{A} + \tilde{Z}/2$ "orthogonal" to $L_\pm$.



Parabolic Ladder Operators

A generator $X = -B + Z/2$ of the subgroup N' gets the equations:

$$b + 2c = \lambda a, \qquad -a = \lambda b, \qquad \frac{a}{2} = \lambda c,$$

which can be resolved if and only if $\lambda^2 = 0$. Restricted with the real (complex) root $\lambda = 0$ make operators $L_\pm = -\tilde{B} + \tilde{Z}/2$. Does not affect eigenvalues and thus are useless. However, a dual number $\lambda_t = t\varepsilon$, $t \in \mathbb{R}$ leads to the operator $L_\pm = \pm t\varepsilon\tilde{A} - \tilde{B} + \tilde{B}/2$, which allow us to build a $\mathfrak{sl}_2$-modules with a one-dimensional continuous(!) chain of eigenvalues.

- K Introduction of complex numbers is a necessity for the *existence* of raising/lowering operators;
- N we need dual numbers to make raising/lowering operators *useful*;
- A double number are required for neither existence nor usability of raising/lowering operators, but do provide an enhancement.

Similarity and Correspondence

Principle of Similarity and correspondence

1. Subgroups K, N and A play the similar role in a structure of the group $SL_2(\mathbb{R})$ and its representations.
2. The subgroups shall be swapped together with the respective replacement of hypercomplex unit ι.

Manifestations:

- The action of $SL_2(\mathbb{R})$ on $SL_2(\mathbb{R})/H$ for $H = A'$, N' or K and linear-fractional transformations of respective numbers.
- Subgroups K, N' and A' and unitary rotations of respective unit cycles.
- Representations induced from subgroup K, N' or A' and unitarity in respective numbers.
- The connection between raising/lowering operators for subgroups K, N' or A' and corresponding numbers.

Bibliography I

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