The Heisenberg group and SL₂(ℝ)
a survival pack

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Phase space, quantum blobs
and squeezed coherent states

▶ Quantum blobs\(^1\) are the smallest phase space units of phase space compatible with the uncertainty principle of quantum mechanics;
▶ Quantum blobs are in a bijective correspondence with the squeezed coherent states\(^2\) from standard quantum mechanics, of which they are a phase space picture;
▶ Quantum blobs have the symplectic group as group of symmetries;
▶ Thus: blobs (aka squeezed states) are a family invariant under quadratic Hamiltonians, which transform them geometrically, that is by a change of variables.

**Question:** Do we have other states transformed geometrically by quadratic Hamiltonians?

**Answer (well-known, yet will be revised):** For the harmonic oscillator any state is moving geometrically in the phase space.

Ladder Operators
and the Hermite functions

Let $P$ and $Q$ satisfy CCR: $[Q, P] = i\hbar I$ of the Weyl algebra $\mathfrak{h}_1$. Consider complexification of $\mathfrak{h}_1$ and define operators:

$$a^\pm = \sqrt{\frac{m\omega}{2\hbar}} \left( Q \mp \frac{i}{m\omega} P \right), \quad \text{then} \quad [a^-, a^+] = 1. \quad (1)$$

For a solution $|0\rangle$ of $a^- |0\rangle = 0$ we define $|n\rangle = (\pi^n n!)^{-1/2} (a^+)^n |0\rangle$, then:

1. $(a^-)^* = a^+$ on $L_2(\mathbb{R})$.
2. The name ladder operators is explained by the diagram:

$$|0\rangle \xleftarrow{a^-} |1\rangle \xleftarrow{a^-} |2\rangle \xleftarrow{a^-} |3\rangle \xleftarrow{a^-} \ldots$$

Since

$$(a^-) |n\rangle = -(\pi n)^{1/2} |n - 1\rangle, \quad (a^+) |n\rangle = (\pi(n + 1))^{1/2} |n + 1\rangle$$

3. Orthonormality: $\langle n|k \rangle = \delta_{nk}$. 

Ladder Operators
and representation theory
The above construction relays on CCR: \([Q, P] = i\hbar I\) or \([a^-, a^+] = 1\) only. No specific realisation is assumed. Conclusions:

- There are different vacuums \(|0\rangle\) (that is \(a^-|0\rangle = 0\)) for different \(m\omega\).
- Any normalised vacuum \(|0\rangle\) creates the orthonormal basis \(\{|n\rangle\}_n\) of an irreducible invariant space.
- Any two irreducible spaces are isomorphic by \(|n\rangle \rightarrow |n\rangle'\).
- Thus, it provides the (constructive!) proof of the Stone-von Neumann theorem on the uniqueness of representation of CCG.

Remark 1.

- There is no genuinely “non-squeezed” states, they are all squeezed in a different way.
- All vacuums are minimising uncertainty \(\Delta Q \cdot \Delta P(\geq \frac{\hbar}{2})\).
- Obviously, this approach inspired Bargmann to produce classification of UIRs of \(SL_2(\mathbb{R})\).
Ladder Operators
and quantum harmonic oscillator

Hamiltonian of the harmonic oscillator.

\[ H = \frac{\hbar \omega}{2} \left( a^+ a^- + a^- a^+ \right) = \hbar \omega (a^+ a^- + \frac{1}{2}) = \frac{1}{2m} p^2 + \frac{m \omega^2}{2} Q^2. \quad (2) \]

Using identities in 2 we obtain spectral decomposition

\[ H \left| n \right\rangle = \hbar \omega (n + \frac{1}{2}) \left| n \right\rangle. \]

1. The spectrum of the harmonic oscillator is discrete.
2. The eigenfunctions are provided by the \( \left| n \right\rangle \).
3. The ladder operators acts on the spectrum by a shift \( \hbar \omega \) due to the commutation relation \([H, a^\pm] = 2a^\pm\):

\[
H(a^+ \left| k \right\rangle) = (a^+ H + 2a^+) \left| k \right\rangle = a^+ (H \left| k \right\rangle) + 2a^+ \left| k \right\rangle = (2k + 1)a^+ \left| k \right\rangle + 2a^+ \left| k \right\rangle = (2k + 3)a^+ \left| k \right\rangle.
\]
Hamiltonian from Ladder Operators
pros and contras

- Representation independent.
- Inspired ladder technique for other Hamiltonians, e.g. hydrogen atom by Schrödinger or resent works in SUSY QM.
- Time evolution of an arbitrary superposition $\sum a_n |n\rangle$ is $\sum e^{-i\omega(n+1/2)t} a_n |n\rangle$.
- In configuration space (the Schrödinger model) the dynamic is a Gauss-type (quadratic Fourier) integral transform.
- There is a specific space—Fock–Segal–Bargmann (FSB) space—which makes the dynamic geometric.
Harmonic oscillator
in FSB representation
Using the displacement operator $D(z) = e^{(\bar{z}a^+ + za^-)/2} = e^{xQ+yP}$, $z = m\omega x + iy$ (the representation of the Heisenberg group, in fact) we create the coherent states $|z\rangle = D(z)|0\rangle$ and the coherent state transform:

$$\mathcal{W}: f \mapsto f(z) = \langle f|z\rangle$$

(3)

The image—the Fock–Segal–Bargmann space—consists of analytic functions on $\mathbb{C}$. Hamiltonian of the harmonic oscillator:

$$H = \frac{1}{2m}(d\tilde{\sigma}_h^x)^2 + \frac{m\omega^2}{2}(d\tilde{\sigma}_h^y)^2$$

(4)

$$= \frac{1}{2m}\partial_{xx}^2 + \frac{m\omega^2}{2}\partial_{yy}^2$$

$$+ i\pi\hbar \left( m\omega^2 x\partial_y - \frac{1}{m} y\partial_x \right) - \pi^2\hbar^2 \left( \frac{m\omega^2}{2} x^2 + \frac{1}{2m} y^2 \right).$$

The oscillator’s dynamics in FSB space is geometric rotation:

$$A_t: f(z) \mapsto e^{-\pi\hbar(i\omega t - m\omega y^2 - x^2/(m\omega))} f(e^{-2\pi i\hbar \omega t}z)$$

despite of the presence of the second derivatives!
Harmonic oscillator
a solution in FSB space

The mystery resolved:\(^3\)
Functions in FSB transform are analytic functions of the variable
\(z = m\omega x + iy\), thus the second order “Laplacian”

\[
\frac{1}{2m} \partial^2_{xx} + \frac{m\omega^2}{2} \partial^2_{yy}
\]

vanishes on FSB space.

The key element:
the representation on the phase space is reducible, giving the room for an additional condition(s), e.g. analyticity, to specify vectors from the irreducible component.

Question: Are there other examples of a geometric dynamic?

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\(^3\) Almalki and Kisil, “Geometric Dynamics of a Harmonic Oscillator, Non-Admissible Mother Wavelets and Squeezed States”, 2018.
The shear Lie algebra
just one nilpotency step away from the Heisenberg–Weyl

Let $\mathfrak{a}$ be the three-step nilpotent Lie algebra whose basic elements
$\{X_1, X_2, X_3, X_4\}$ with the following non-vanishing commutators$^4$: $^5$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$  \hspace{1cm} (5)

Obviously, $\{X_1, X_3, X_4\}$ is the Heisenberg–Weyl algebra. We will
systematically employ this inclusion to save our calculations.
The algebra $\mathfrak{a}$ and respective Lie group—the shear group (aka quartic
$^6$ or Engel group$^7$)—is a toy model to try any generalisations$^8$. $^9$

$^4$Corwin and Greenleaf, *Representations of Nilpotent Lie Groups and Their


$^7$Ardentov and Sachkov, “Maxwell Strata and Cut Locus in the Sub-Riemannian

$^8$Howe, Ratcliff, and Wildberger, “Symbol Mappings for Certain Nilpotent Groups”,
1984.

$^9$I. Beltiţă, D. Beltiţă, and Pascu, “Boundedness for Pseudo-Differential Calculus on
Nilpotent Lie Groups”, 2013.
The shear group
including the Heisenberg group

The corresponding Lie group is a three-step nilpotent $\mathbb{A}$ and the group law is given by:

\[(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2, x_4 + y_4 + x_1 y_3 + \frac{1}{2} x_1^2 y_2),\]

where $x_j, y_j \in \mathbb{R}$ and the canonical coordinates are $(x_1, x_2, x_3, x_4) := \exp(x_4 X_4) \exp(x_3 X_3) \exp(x_2 X_2) \exp(x_1 X_1)$. The identity element is $(0, 0, 0, 0)$ and the inverse of an element $(x_1, x_2, x_3, x_4)$ is

\[(-x_1, -x_2, x_1 x_2 - x_3, x_1 x_3 - \frac{1}{2} x_1^2 x_2 - x_4).\]

The group centre is

\[Z(\mathbb{A}) = \{(0, 0, 0, x_4) \in \mathbb{A} : x_4 \in \mathbb{R}\}.\]

The Heisenberg group $\mathbb{H}$ is isomorphic to the subgroup $\tilde{\mathbb{H}} = \{(x_1, 0, x_3, x_4) \in \mathbb{A} : x_j \in \mathbb{R}\}$ by $(x, y, s) \mapsto (x, 0, y, s), (x, y, s) \in \mathbb{H}$. 

\[\text{University of Leeds}\]
The shear group and Schrödinger group

The Schrödinger group is the semi-direct product $S = \mathbb{H} \rtimes_A SL_2(\mathbb{R})$, where $SL_2(\mathbb{R})$ is the group of all $2 \times 2$ real matrices with the unit determinant. The action $A$ of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by:\(^{10}\)

$$A(g) : (x, y, s) \mapsto (ax + by, cx + dy, s),$$

(8)

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $(x, y, s) \in \mathbb{H}$. Let

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \ x \in \mathbb{R} \right\}$$

be the subgroup of $SL_2(\mathbb{R})$, it is easy to check that $A$ is isomorphic to the subgroup $\mathbb{H} \rtimes_A N$ of $S$ through the map:

$$(x_1, x_2, x_3, x_4) \mapsto ((x_1, x_3, x_4), n(x_2)) \in \mathbb{H} \rtimes_A N,$$

where $(x_1, x_3, x_4) \in \mathbb{H}$ and $n(x_2) := \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$.

\(^{10}\)M. A. d. Gosson, Symplectic Methods in Harmonic Analysis and in Mathematical Physics, 2011; Folland, Harmonic Analysis in Phase Space, 1989.
The shear group and Schrödinger group

Geometrically: *shear transform* with the angle $\tan^{-1} x_2$:

$$n(x_2)(x_1, x_3) := \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 x_1 + x_3 \end{pmatrix}. \quad (9)$$

Physically, for a particle with coordinate $x_3$ and the constant velocity $x_1$: after a period of time $-x_2$ the particle will still have the velocity $x_1$ but its new coordinate will be $x_3 - x_2 x_1$. 
UIR of the shear group
induction and the Kirillov’s orbit method

Classification of UIRs of the shear group is nicely accomplished by the Kirillov’s orbit method (by Kirillov himself\textsuperscript{11} :-) and UIRs are explicitly constructed by the Mackey induction procedure. Avoiding details, the main set of UIRs is parametrised by by two “Planck constants” $\hbar_2$ and $\hbar_4$ and induced by the character $\chi_{\hbar_2 \hbar_2}(0, x_2, x_3, x_4) = e^{2\pi(\hbar_2 x_2 + \hbar_4 x_4)}$ of the maximal abelian sungroup $H_a = \{(0, x_2, x_3, x_4)\}$ in $L_2(\mathbb{R})$ is:\textsuperscript{12}

\[ [\rho_{\hbar_2 \hbar_4}(x_1, x_2, x_3, x_4)f](x_1') = e^{2\pi i(\hbar_2 x_2 + \hbar_4(x_4 - x_3 x_1' + \frac{1}{2} x_2 x_1'^2))}f(x_1' - x_1). \] (10)

This representation is irreducible since its restriction to the Heisenberg group $\tilde{H}$ coincides with the irreducible Schrödinger representation with $\hbar_4$ being the Planck constant.

\textsuperscript{11}Kirillov, \textit{Lectures on the Orbit Method}, 2004, § 3.2.
\textsuperscript{12}Ibid., 2004, § 3.3, (19).
Coherent state transform

aka voice transform, wavelet transform, etc.

For a \( G \), \( \rho \) and a fixed mother wavelet \( \phi \in H \), the wavelet transform\(^{13}\) is:

\[
[W^\rho_\phi f](g) = \langle \rho(g^{-1})f, \phi \rangle = \langle f, \rho(g)\phi \rangle, \quad g \in G.
\]

Let a mother wavelet \( \phi \) be a joint eigenvector of \( \rho(h) \) for all \( h \in H \):

\[
\rho(h)\phi = \chi(h)\phi \quad \text{for all} \quad h \in H. \tag{11}
\]

and a character \( \chi \) of \( H \). Then

\[
[W^\rho_\phi f](gh) = \overline{\chi}(h)[W^\rho_\phi f](g). \tag{12}
\]

Thus the restriction of the left regular representation \( \Lambda \) (intertwined with \( \rho \) by \( W^\rho_\phi \)) is induced by \( \chi \). For a section \( s : G/H \rightarrow G \), and \( \phi \) satisfying (11), \textit{induced wavelet transform}\(^{14}\) \( W^\rho_\phi \) is

\[
[W^\rho_\phi f](x) = \langle f, \rho(s(x))\phi \rangle, \quad x \in G/H. \tag{13}
\]


Coherent state transform

for the shear group

For $H = \{(0, 0, 0, x_4) \in \mathbb{A} : x_4 \in \mathbb{R}\}$ and the character

$\chi(0, 0, 0, x_4) = e^{2\pi i \hbar_4 x_4}$

any function $\phi \in L^2(\mathbb{R})$ satisfies the eigenvector property (11). Thus, for the respective homogeneous space $\mathbb{A}/\mathbb{Z} \sim \mathbb{R}^3$ and the section $s : \mathbb{A}/\mathbb{Z} \to \mathbb{A}; \ s(x_1, x_2, x_3) = (x_1, x_2, x_3, 0)$ the induced wavelet transform is:

$$[\mathcal{W}_\phi f](x_1, x_2, x_3) = \langle f, \rho_{\hbar_2 \hbar_4}(s(x_1, x_2, x_3)) \phi \rangle$$

$$= \int_{\mathbb{R}} f(y) e^{-2\pi i (\hbar_2 x_2 + \hbar_4 (-x_3 y + \frac{1}{2} x_2 y^2))} \overline{\phi}(y - x_1) \, dy$$

$$= e^{-2\pi i \hbar_2 x_2} \int_{\mathbb{R}} f(y) e^{-2\pi i \hbar_4 (-x_3 y + \frac{1}{2} x_2 y^2)} \overline{\phi}(y - x_1) \, dy.$$

For the Heisenberg group ($x_2 = 0$) it is Fourier–Wigner transform.\textsuperscript{15}

For the share group it is the quadratic Fourier transform or the Gauss integral transform.\textsuperscript{16}

\textsuperscript{15}Folland, *Harmonic Analysis in Phase Space*, 1989.

CST on the shear group
non-square-integrability, is it an issue?
For a fixed unit vector $\phi \in L_2(\mathbb{R})$, let $L_\phi(\mathbb{A}/\mathbb{Z})$ be the image space of the wavelet transform $W_\phi$ (14) equipped with the family of inner products parametrised by $x_2 \in \mathbb{R}$

$$\langle u, v \rangle_{x_2} := \int_{\mathbb{R}^2} u(x_1, x_2, x_3) \overline{v(x_1, x_2, x_3)} h_4 \, dx_1 \, dx_3 .$$

(15)

The respective norm is denoted by $\| u \|_{x_2}$. $W_\phi$ is unitary map $H \rightarrow L_\phi(\mathbb{A}/\mathbb{Z}), \| \cdot \|_{x_2}$, furthermore we have the orthogonality relation:

$$\langle W_{\phi_1} f_1, W_{\phi_2} f_2 \rangle_{x_2} = \langle f_1, f_2 \rangle \langle \phi_1, \phi_2 \rangle$$

for any $x_2 \in \mathbb{R}$.

(16)

Then its adjoint:

$$[M_\phi (x_2)f](t) = \int_{\mathbb{R}^2} f(x_1, x_2, x_3) \rho(x_1, x_2, x_3, 0) \phi(t) h_4 \, dx_1 \, dx_3 .$$

(17)

is its inverse: $M_\psi (x_2) \circ W_\phi = \langle \psi, \phi \rangle I$ if $\langle \psi, \phi \rangle = 1$. 
Characterisation of CST image and Lie derivatives

For the right shift $R(g) : f(g') \mapsto f(g'g)$, the covariant transform intertwines $R(g)$ with the action $\rho$ on vacuum states:

$$R(g) \circ \mathcal{W}_\phi = \mathcal{W}_{\rho(g)\phi}. \quad (\text{cf. } \Lambda(g) \circ \mathcal{W}_\phi = \mathcal{W}_\phi \circ \rho(g)). \quad (18)$$

Let $\rho$ be a UIR of $G$, which can be extended by integration to a vector space $V$ of functions or distributions on $G$. If $\phi \in H$ satisfy the

$$\rho(a)\phi = 0, \quad \text{where } \rho(a)\phi = \int_G a(g) \rho(g)\phi \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{\nu}(g) = \langle \nu, \rho(g)\phi \rangle$ obeys the condition:

$$R(\bar{a})\nu = 0, \quad \text{where } R(\bar{a}) = \int_G \bar{a}(g) R(g) \, dg, \quad (19)$$

with $R$ being the right regular representation of $G$.

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The shear group CST

image characterisation: structural condition

The derived representations of the basis \( \{ X_1, X_2, X_3, X_4 \} \) of \( a \) are:

\[
\begin{align*}
\rho_{h_2 h_4}^{X_1} &= -\frac{d}{dy}; \\
\rho_{h_2 h_4}^{X_2} &= 2\pi i h_2 + \pi i h_4 y^2; \\
\rho_{h_2 h_4}^{X_3} &= -2\pi i h_4 y; \\
\end{align*}
\]  

(20)

Lie derivative \( L^X \) is the derived right regular representation:

\[
\begin{align*}
L^X_{X_1} &= \partial_1; \\
L^X_{X_2} &= \partial_2 + x_1 \partial_3 - i\pi h_4 x_1^2 I; \\
L^X_{X_3} &= \partial_3 - 2\pi i h_4 x_1 I; \\
\end{align*}
\]  

(21)

Any function \( \phi \) satisfies the relation for the derived representation

\[
\left( (\rho_{h_2 h_4}^{X_3})^2 - 4\pi i h_4 \rho_{h_2 h_4}^{X_2} - 8\pi^2 h_2 h_4 I \right) \phi = 0.
\]

The image \( f \in L_{\phi}(\mathbb{A}/\mathbb{Z}) \) of the wavelet transform \( \mathcal{W}_{\phi} \) is annihilated by the respective Lie derivatives operator \( S f = 0 \) where

\[
S = (L^X_{X_3})^2 + 4\pi i h_4 L^X_{X_2} - 8\pi^2 h_2 h_4 I
\]

(22)

\[
= \partial_{33}^2 + 4\pi i h_4 \partial_2 - 8\pi^2 h_2 h_4 I.
\]

This will be called the *structural condition* because it is determined by the structure of the particular representation \( \rho_{h_2 h_4} \) (10).
The shear group CST
image characterisation: the Gaussian

A particular choice of a mother wavelet $\phi$ such that $\phi$ lies in $L^2(\mathbb{R})$ and is a null solution to the “first order” operator, cf. (20):

$$d\rho_{\hbar_2\hbar_4}^{X_1+ax_2+i\mathcal{E}x_3} = d\rho_{\hbar_2\hbar_4}^{X_1} + a d\rho_{\hbar_2\hbar_4}^{x_2} + i\mathcal{E} d\rho_{\hbar_2\hbar_4}^{x_3},$$

where $a$ and $\mathcal{E}$ some real constants. It is clear that, the function

$$\phi(y) = \exp \left( \frac{\pi ia\hbar_4}{3} y^3 + \pi \mathcal{E}\hbar_4 y^2 + 2\pi ia\hbar_2 y \right),$$

is a generic solution and square integrability of $\phi$ requires that $\mathcal{E}\hbar_4$ is strictly negative. Furthermore, for the purpose of this work it is sufficient to use the simpler mother wavelet corresponding to the value $a = 0$:

$$\phi(y) = e^{\pi \mathcal{E}\hbar_4 y^2}, \quad \hbar_4 > 0, \quad \mathcal{E} < 0. \quad (24)$$

If $a \neq 0$ then we obtain Airy beam decomposition (cubic Fourier transform).\(^{19}\)

\(^{19}\)Torre, “A Note on the Airy Beams in the Light of the Symmetry Algebra Based Approach”, 2009.
The shear group CST
image characterisation: the analytic condition

Any function $f$ in $L_\phi(A/Z)$ for $\phi$ (24) satisfies $\mathcal{C}f = 0$ for the partial differential operator produced from (23) with $a = 0$:

$$\mathcal{C} = (\mathcal{L}^{X_1} - iE\mathcal{L}^{X_3}) = \partial_1 - iE\partial_3 - 2\pi\hbar_4 E x_1. \quad (25)$$

By *peeling* (multiplication with a suitable factor) it can be converted into the Cauchy–Riemann equation, we call (25) the analyticity condition. The CST (14) with the Gaussian for the Haisenberg group ($x_2 = 0$) becomes the Fock–Segal–Bargmann transform to analytic function of $z = -Ex_1 + ix_3$.

Since the structural operator $S = \partial_{33}^2 + 4\pi i\hbar_4 \partial_2 - 8\pi^2 \hbar_2 \hbar_4 I$ (22) is a Schrödinger equation of a free particle (with $x_2$ being time) we get

**Physical characterisation of $L_\phi(A/Z)$:** consists of wavefunctions expanded from the phase space $\mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}$ by free time-evolution.
Geometric dynamics of HO
from the Heisenberg group

The harmonic oscillator with mass $m$ and frequency $\omega$ is quantised

$$ H = \frac{1}{2m} (id\tilde{\sigma}_h^{X_1})^2 + \frac{m\omega^2}{2} (id\tilde{\sigma}_h^{X_3})^2 $$

$$ = -\frac{1}{2m} \partial_{11}^2 - \frac{m\omega^2}{2} \partial_{22}^2 + \frac{2\pi i}{m} x_3 \partial_1 + \frac{2\pi^2 \hbar^2}{m} x_3^2 $$

(26)

Our aim is dynamics in geometric terms by lowering the order of the differential operator (26) using the analyticity condition:

$$ \tilde{H} = H + (A\partial_1 + iB\partial_3 + CI)(\mathcal{L}^{X_1} - iE\mathcal{L}^{X_3}) $$

This requires $A = \frac{1}{2m}$, $B = -\frac{1}{2} \omega$, $C = -\pi\hbar\omega x_1$ and $E = -m\omega$.

$$ \tilde{H} = \frac{2\pi i}{m} x_3 \partial_1 - 2\pi i \hbar m\omega^2 x_1 \partial_3 + \left( \pi\hbar\omega + \frac{2\pi^2 \hbar^2}{m} (x_3^2 - m^2 \omega^2 x_1^2) \right) I. $$

(27)

The solution is given by the rotation of the complex variable $x_3 - im\omega x_1$.

New observation: the value of $E$ is uniquely defined and the corresponding vacuum vector $\phi(q) = e^{\pi\hbar E q^2} = e^{-\pi\hbar m\omega q^2}$ is fixed.
Geometric dynamics of HO from the shear group

Similarly, the Hamiltonian of HO for the shear group \((x_2 \neq 0)\) is

\[
H = \left( \frac{1}{2m} (\text{id} \tilde{\rho} X_1^2) + \frac{m \omega^2}{2} (\text{id} \tilde{\rho} X_3^2) \right)
\]
\[
= -\frac{1}{2m} \partial_{11}^2 - \frac{1}{2m} x_2^2 \partial_{33}^2 - \frac{m \omega^2}{2} \partial_{33}^2 - \frac{1}{m} x_2 \partial_{13}^2
\]
\[
+ \frac{2\pi \hbar}{m} x_3 \partial_1 + \frac{2\pi \hbar}{m} x_2 x_3 \partial_3 - \frac{1}{m} \left( -\pi \hbar x_2 - 2\pi^2 \hbar^2 x_3^2 \right) I
\]

(28)

Adjusting by the analytic (25) and structural (22) conditions:

\[
H_1 = H + (A \partial_1 + B \partial_2 + C \partial_3 + KI)C + FS.
\]

To eliminate all second order derivatives take \(A = \frac{1}{2m}, B = 0, C = \frac{1}{m} (\frac{i}{2} E + x_2), K = \frac{\pi \hbar}{m} x_1 (E + 2ix_2)\). and \(F = -\frac{1}{2m} (ix_2 - E)^2 + \frac{m \omega^2}{2}\).

**Difference with the Heisenberg group:** there is no restrictions for the parameter \(E\), any squeezed states\(^{20-21} e^{\pi \hbar E y^2}, E < 0\) can be mother wavelet.

\(^{20}\text{Gazeau, Coherent States in Quantum Physics, 2009.}\)
Geometric dynamics of HO on the shear group solution

The adjusted Hailtonian is:

\[ H_1 = \frac{2\pi i \hbar_4}{m} \left( (x_3 + x_1 x_2) \partial_1 - ((i x_2 - E)^2 - m^2 \omega^2) \partial_2 - (E^2 x_1 - x_2 x_3) \partial_3 \right) \]

\[ - \frac{\pi \hbar_4}{m} \left( 8i \pi \hbar_2 E x_2 - i x_2 + 4 \pi \hbar_2 x_2^2 - 2 \pi \hbar_4 x_3^2 ight. \]

\[ + 4 \pi \hbar_2 m^2 \omega^2 + E - 4 \pi \hbar_2 E^2 + 4i \pi E \hbar_4 x_1^2 x_2 + 2 \pi \hbar_4 E^2 x_1^2 \) \right) \]

I.

Using the analyticity condition in the variable \( z = x_3 + i E x_1 \):

\[ f(t, x_1, x_2, x_3) = \frac{\sqrt{E + m \omega}}{\sqrt{i x_2 + E + m \omega}} \times \exp \left( i \pi \omega t - \pi \hbar_4 E x_1^2 - 2 \pi \hbar_2 x_2 - \pi \hbar_4 \frac{(x_3 - i E x_1)^2}{i x_2 + E + m \omega} \right) \]

\[ \times f_1 \left( e^{2i \pi \omega t} \frac{x_3 - i E x_1}{i x_2 + E + m \omega}, e^{4i \pi \omega t} \frac{m \omega - (i x_2 + E)}{m \omega + (i x_2 + E)} \right). \]

where \( f_1(z, u) \) is an arbitrary function of two variables such that:

- analytic in the first variable.
- solves heat-like eqn. \( \partial_u f_1(z, u) = -\frac{1}{8 \pi \hbar_4 m \omega} \partial_{zz}^2 f_1(z, u) \)
The solution: bounds for a possible squeeze

Figure: Shear parameter and analytic continuation. The solid circle is the image of the line $ix_2 + E$ under the Cayley transformation. The shadowed region (the annulus with radii $c$ and 1) is obtained from the solid circle under rotation around the origin. The dashed circle of the radius $R$ bounds the domain of the analytic continuation of the heat equation solution.

Left: $E = \omega$ (thus $c = 0$)—there always exists a part of the shaded region inside the circle of a radius $R$ (even for $R = 0$).

Middle: some $E$ within the bound—there is a thick arc inside of the dashed circle, the arc corresponds to values of $x_2$ with a meaningful solution (29).

Right: a shear parameter $E$ is outside of the range, a state is squeezed too much, no values of $x_2$ are allowed in (29).


Bibliography V


