Integrability of the Spin System

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I. Review

The nature of (1+1)-dimensional integrable systems is now well understood [1].

Nonlinear Schrödinger equation (NLSE)

$$i\varphi_t + \varphi_{xx} + 2|\varphi|^2\varphi = 0 \tag{1}$$

with boundary condition

$$\varphi(x,t)|_{|x|\to\infty} \to 0, \tag{2}$$

where $\varphi(x,t)$ is a complex-valued function (classical charged field), subscripts mean the partial derivatives of the corresponding variables.

[1] M.J. Ablowitz and P.A. Clarkson, Solitons, Non-linear Evolution Equations and Inverse Scattering (Cambridge University Press, Cambridge, 1992).

The integrability of the NLSE (1) through the IST is realized by the following Lax pair:

$$\Phi_x = U\Phi, \tag{3a}$$

$$\Phi_t = V\Phi, \tag{3b}$$

where

$$U = \lambda U_1 + U_0. \tag{4a}$$

Here

$$U_1 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & i\bar{\varphi} \\ i\varphi & 0 \end{pmatrix}.$$

$$V = \lambda^2 V_2 + \lambda V_1 + V_0, \tag{4b}$$

with

$$V_2 = -U_1, \quad V_1 = -U_0, \quad V_0 = \begin{pmatrix} -i|\varphi|^2 & \bar{\varphi}_x \\ -\varphi_x & i|\varphi|^2 \end{pmatrix}.$$

An interesting subclass of integrable systems, useful both from the mathematical and physical points of view, is the set of integrable spin systems.
(1+1)-dimensional isotropic classical continuous Heisenberg ferromagnet model (HFM):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \tag{5}$$

with boundary condition

$$\mathbf{S}(S_1, S_2, S_3)|_{x\to\infty} \to (0, 0, \pm 1), \tag{6}$$

where $\mathbf{S}(x, t)$ is a spin vector, $\times$ means a vector product. The range of the value of $\mathbf{S}$ is a subset of the unit sphere in R^3.

The integrability of the HFM (5) using the IST problem is associated with the compatibility condition of the system

$$\Phi_x = U\Phi, \tag{7a}$$

$$\Phi_t = V\Phi, \tag{7b}$$

where

$$U = \frac{i}{2}\lambda S, \quad V = \frac{i\lambda^2}{2}S + \frac{\lambda}{4}[S, S_x]. \tag{8}$$

Since the identification of the first integrable Heisenberg spin systems [2,3], several other integrable spin systems in (1+1)-dimensional have been identified and investigated through geometrical and gauge equivalence concepts and its the IST method.

[2] M. Lakshmanan, Phys. Lett. A 61 (1977) 53.
[3] L.A. Takhtajan, Phys. Lett. A 64 (1977) 235.

Integrable spin systems in (2+1)-dimensions

The equation (5) admits a series of integrable (2+1)-dimensional generalizations. One of them is the following equation:

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xy} + u\mathbf{S}_x, \tag{9a}$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y). \tag{9b}$$

Line system for eq.(9)

$$\Phi_{1x} = U_1 \Phi_1, \tag{10a}$$

$$\Phi_{1t} = \beta\lambda\Phi_{1y} + V_1\Phi_1, \tag{10b}$$

where

$$U_1 = \frac{i}{2}\lambda S, \tag{11a}$$

$$V_1 = \alpha\left(\frac{i\lambda^2}{2}S + \frac{\lambda}{4}[S, S_x]\right) + \beta\frac{\lambda}{4}\left([S, S_y] + 2iuS\right). \tag{11b}$$

The integrable spin equation (9) is investigated in [4]. It is shown that its geometrical and gauge equivalent counterparts are the (2+1)-dimensional non-linear Schrödinger equation belonging to the class of equations discovered by Calogero and then discussed by Zakharov and studied by Strachan. It has the form

$$iq_t - \alpha q_{xx} - \beta q_{xy} - vq = 0, \quad (12a)$$

$$ip_t + \alpha p_{xx} + \beta p_{xy} + vp = 0, \quad (12b)$$

$$v_x = 2[\alpha(pq)_x + \beta(pq)_y], \quad (12c)$$

where α and β - real constants, q and p - complex-valued functions, v is a potential.

[4] R. Myrzakulov, S. Vijayalakshmi, G. Nugmanova, M. Lakshmanan. A (2+1)-dimensional integrable spin model: Geometrical and gauge equivalent counterpart, solitons and localized coherent structures. Phys. Lett. A 233 (1997) 391-396.

[5] Chen Chi, Zhou Zi-Xiang. *Darboux Transformation and Exact Solutions of the Myrzakulov-I Equation*, Chinese Physics Letters, v26, N8, 080504 (2009)

[6] Chen Hai, Zhou Zi-Xiang. *Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation*, Chinese Physics Letters., v31, N12, 120504 (2014)

[7] Chen Hai, Zhou Zi-Xiang. *Global explicit solutions with n double spectral parameters for the Myrzakulov-I equation*, Modern Physics Letters B, v30, N29, 1650358 (2016)

II. Results

Integrable spin system with self-consistent potentials

The integrable Heisenberg ferromagnetic equation reads as

$$iS_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (1)$$

$$iW_x + \omega[S, W] = 0, \quad (2)$$

where $S = S_i\sigma_i$, $W = W_i\sigma_i$, $S^2 = I$, $W^2 = b(t)I$, $b(t) = const(t)$, $I = diag(1, 1)$, $[A, B] = AB - BA$, ω is a real constant and σ_i are Pauli matrices.

The Lax representation can be written in the form

$$\Phi_x = U\Phi, \quad (3)$$

$$\Phi_t = V\Phi, \quad (4)$$

where the matrix operators U and V have the form

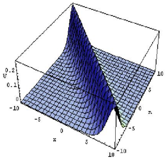
$$U = -i\lambda S, \quad (5)$$

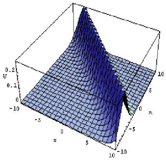
$$V = \lambda^2 V_2 + \lambda V_1 + \left(\frac{i}{\lambda+\omega} - \frac{i}{\omega}\right) W. \quad (6)$$

Here

$$V_2 = -2iS, \quad V_1 = 0.5[S, S_x], \quad (7)$$

$$S = \begin{pmatrix} S_3 & S^+ \\ S^- & -S_3 \end{pmatrix}, \quad W = \begin{pmatrix} W_3 & W^+ \\ W^- & -W_3 \end{pmatrix}, \quad (8)$$

[a] 

[b] 

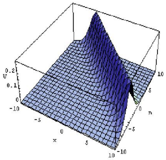
[c] 

Figure: One-soliton solution

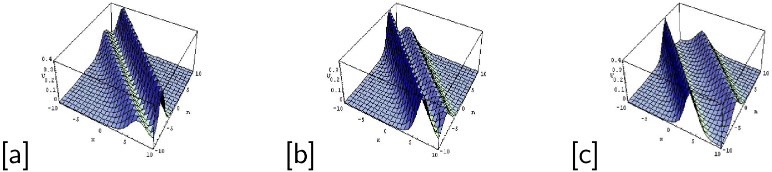


Figure: The interaction of two solitons

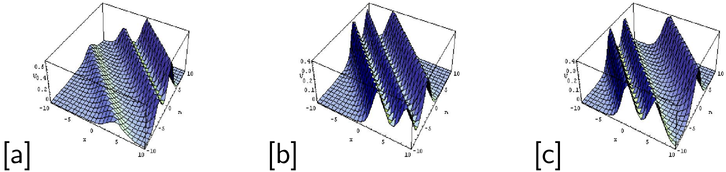


Figure: The interaction of three solitons

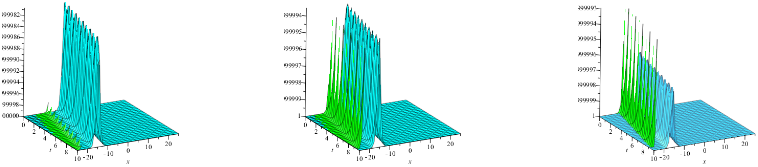


Figure: The relationship between the spin vector and the vector potential

1-layer spin system

Consider the spin vector $\mathbf{A} = (A_1, A_2, A_3)$, where $\mathbf{A}^2 = 1$. Let this spin vector obey the 1-layer spin system which reads as

$$\mathbf{A}_t + \mathbf{A} \wedge \mathbf{A}_{xx} + u_1 \mathbf{A}_x + \mathbf{F} = 0, \tag{9}$$

where $u_1(x, t, A_j, A_{jx})$ is the potential, $\mathbf{F}$ is some vector function. The matrix form of the spin system looks like

$$iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + F \quad = \quad 0, \tag{10}$$

where

$$A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I = diag(1, 1), \quad A^{\pm} = A_1 \pm iA_2. \tag{11}$$

$$F = \begin{pmatrix} F_3 & F^- \\ F^+ & -F_3 \end{pmatrix}, \quad F^{\pm} = F_1 \pm iF_2. \tag{12}$$

We consider the following particular case of the spin system

$$\mathbf{A}_t + \mathbf{A} \wedge \mathbf{A}_{xx} + u_1 \mathbf{A}_x + v_1 \mathbf{H} \wedge \mathbf{A} = 0, \tag{13}$$

where $v_1(x, t, A_j, A_{jx})$ is the potential, $\mathbf{H} = (0, 0, 1)$ is the constant magnetic field. It is interesting to note that the integrable 2-layer spin system contains constant magnetic field $\mathbf{H}$. It seems that this constant magnetic vector plays an important role in theory of "integrable multilayer spin system" and in nonlinear dynamics of magnetic systems.

Geometrical equivalent counterpart

Let us find the geometrical equivalent counterpart of the 1-layer spin system (13). To do that, consider 3-dimensional curve in R^3. This curve is given by the following vectors $\mathbf{e}_k$. These vectors satisfy the following equations

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \qquad (14)$$

Here $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the unit tangent, normal and binormal vectors to the curve. The matrices C and G have the forms

$$C = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \quad (15)$$

The curvature and torsion of the curve are given by the following formulas

$$k_1 = \sqrt{\mathbf{e}_{1x}^2}, \quad \tau_1 = \frac{\mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1xx})}{\mathbf{e}_{1x}^2}. \tag{16}$$

The compatibility condition of the equations (14) is given by

$$C_t - G_x + [C, G] = 0, \tag{17}$$

or in elements

$$k_{1t} = \omega_{3x} + \tau_1 \omega_2, \tag{18}$$

$$\tau_{1t} = \omega_{1x} - k_1 \omega_2, \tag{19}$$

$$\omega_{2x} = \tau_1 \omega_3 - k_1 \omega_1. \tag{20}$$

Now we do the following identifications:

$$\mathbf{A} \equiv \mathbf{e}_1, \quad \mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3. \tag{21}$$

Then we have

$$k_1^2 = \mathbf{A}_x^2, \tag{22}$$

$$\tau_1 = \frac{\mathbf{A} \cdot (\mathbf{A}_x \wedge \mathbf{A}_{xx})}{\mathbf{A}_x^2}, \tag{23}$$

and

$$\omega_1 = -\frac{k_{1xx} + F_2\tau_1 + F_{3x}}{k_1} + (\tau_1 - u_1)\tau_1, \tag{24}$$

$$\omega_2 = k_{1x} + F_3, \tag{25}$$

$$\omega_3 = k_1(\tau_1 - u_1) - F_2, \tag{26}$$

with $F_1 = E_1 = 0$. The equations for k_1 and τ_1 reads as

$$k_{1t} = 2k_{1x}\tau_1 + k_1\tau_{1x} - (u_1 k_1)_x - F_{2x} + F_3\tau_1, \quad (27)$$

$$\tau_{1t} = \left[-\frac{k_{1xx} + F_2\tau_1 + F_{3x}}{k_1} + (\tau_1 - u_1)\tau_1 - \frac{1}{2}k_1^2\right]_x - F_3 k_1 \quad (28)$$

Next we introduce a new complex function as

$$q_1 = \frac{\kappa_1}{2} e^{-i\partial_x^{-1}\tau_1}. \quad (29)$$

This function satisfies the following equation

$$iq_{1t} + q_{1xx} + 2|q_1|^2 q_1 + \ldots = 0. \quad (30)$$

It is the desired geometrical equivalent counterpart of the spin system (9). If $u_1 = v_1 = 0$, it turns to the NLSE

$$iq_{1t} + q_{1xx} + 2|q_1|^2 q_1 = 0. \quad (31)$$

2-layer spin system

Now we consider two spin vectors $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$, where $\mathbf{A}^2 = \mathbf{B}^2 = 1$. Let these spin vectors satisfy the following 2-layer spin system or the coupled spin system

$$\mathbf{A}_t + \mathbf{A} \wedge \mathbf{A}_{xx} + u_1 \mathbf{A}_x + 2v_1 \mathbf{H} \wedge \mathbf{A} = 0, \quad (32)$$

$$\mathbf{B}_t + \mathbf{B} \wedge \mathbf{B}_{xx} + u_2 \mathbf{B}_x + 2v_2 \mathbf{H} \wedge \mathbf{B} = 0, \quad (33)$$

or in matrix form

$$iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1[\sigma_3, A] = 0, \quad (34)$$

$$iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2[\sigma_3, B] = 0, \quad (35)$$

where $\mathbf{H} = (0, 0, 1)^T$ is the constant magnetic field, u_j and v_j are coupling potentials.

The geometrical equivalent counterpart

In this subsection we present the geometrical equivalent counterpart of the 2-layer spin systems (32)-(33). Now we consider two interacting 3-dimensional curves in R^n. These curves are given by the following two basic vectors $\mathbf{e}_k$ and $\mathbf{l}_k$. The motion of these curves is defined by the following equations

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = D \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \qquad (36)$$

and

$$\begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix}_x = L \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix}_t = N \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix}. \qquad (37)$$

Here $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the unit tangent, normal and binormal vectors respectively to the first curve, $\mathbf{l}_1, \mathbf{l}_2$ and $\mathbf{l}_3$ are the unit tangent, normal and binormal vectors respectively to the second curve, x is the arclength parametrising these both curves. The matrices C, D, L, N are given by

$$C = \begin{pmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{38}$$

$$L = \begin{pmatrix} 0 & k_2 & 0 \\ -k_2 & 0 & \tau_2 \\ 0 & -\tau_2 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}. \tag{39}$$

For the curvatures and torsions of curves we obtain

$$k_1 = \sqrt{\mathbf{e}_{1x}^2}, \quad \tau_1 = \frac{\mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1xx})}{\mathbf{e}_{1x}^2}, \tag{40}$$

$$k_2 = \sqrt{\mathbf{l}_{1x}^2}, \quad \tau_2 = \frac{\mathbf{l}_1 \cdot (\mathbf{l}_{1x} \wedge \mathbf{l}_{1xx})}{\mathbf{l}_{1x}^2}. \tag{41}$$

The equations (36) and (37) are compatible if

$$C_t - G_x + [C, G] = 0, \tag{42}$$

$$L_t - N_x + [L, N] = 0. \tag{43}$$

In elements these equations take the form

$$k_{1t} = \omega_{3x} + \tau_1\omega_2, \tag{44}$$
$$\tau_{1t} = \omega_{1x} - k_1\omega_2, \tag{45}$$
$$\omega_{2x} = \tau_1\omega_3 - k_1\omega_1, \tag{46}$$

and

$$k_{2t} = \theta_{3x} + \tau_2\theta_2, \tag{47}$$
$$\tau_{2t} = \theta_{1x} - k_2\theta_2, \tag{48}$$
$$\theta_{2x} = \tau_2\theta_3 - k_2\theta_1. \tag{49}$$

Our next step is the following identifications:

$$\mathbf{A} \equiv \mathbf{e}_1, \quad \mathbf{B} \equiv \mathbf{l}_1. \tag{50}$$

We also assume that

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3, \quad \mathbf{E} = E_1\mathbf{l}_1 + E_2\mathbf{l}_2 + E_3\mathbf{l}_3, \tag{51}$$

where

$$\mathbf{F} = 2v_1\mathbf{H} \wedge \mathbf{A}, \quad \mathbf{E} = 2v_2\mathbf{H} \wedge \mathbf{B}. \tag{52}$$

Then we obtain

$$k_1^2 = \mathbf{A}_x^2, \tag{53}$$

$$\tau_1 = \frac{\mathbf{A} \cdot (\mathbf{A}_x \wedge \mathbf{A}_{xx})}{\mathbf{A}_x^2}, \tag{54}$$

$$k_2^2 = \mathbf{B}_x^2, \tag{55}$$

$$\tau_2 = \frac{\mathbf{B} \cdot (\mathbf{B}_x \wedge \mathbf{B}_{xx})}{\mathbf{B}_x^2}, \tag{56}$$

and

$$\omega_1 = -\frac{k_{1xx} + F_2\tau_1 + F_{3x}}{k_1} + (\tau_1 - u_1)\tau_1, \quad (57)$$

$$\omega_2 = k_{1x} + F_3, \quad (58)$$

$$\omega_3 = k_1(\tau_1 - u_1) - F_2, \quad (59)$$

$$\theta_1 = -\frac{k_{2xx} + E_2\tau_2 + E_{3x}}{k_2} + (\tau_2 - u_2)\tau_2, \quad (60)$$

$$\theta_2 = k_{2x} + E_3, \quad (61)$$

$$\theta_3 = k_2(\tau_2 - u_2) - E_2. \quad (62)$$

with

$$F_1 = E_1 = 0. \quad (63)$$

We now can write the equations for k_j and τ_j. They look like

$$\mathrm{k}_{1t} = 2k_{1x}\tau_1 + k_1\tau_{1x} - (u_1k_1)_x - F_{2x} + F_3\tau_1, \tag{64}$$

$$\tau_{1t} = \left[-\frac{k_{1xx}+F_2\tau_1+F_{3x}}{k_1} + (\tau_1 - u_1)\tau_1 - \frac{1}{2}k_1^2\right]_x - F_3k_1, \tag{65}$$

$$\mathrm{k}_{2t} = 2k_{2x}\tau_2 + k_2\tau_{2x} - (u_2k_2)_x - E_{2x} + E_3\tau_2, \tag{66}$$

$$\tau_{2t} = \left[-\frac{k_{2xx}+E_2\tau_2+E_{3x}}{k_2} + (\tau_2 - u_2)\tau_2 - \frac{1}{2}k_2^2\right]_x - E_3k_2. \tag{67}$$

Let us now introduce new four real functions α_j and β_j as

$$\alpha_1 = 0.5k_1\sqrt{1+\zeta_1}, \tag{68}$$

$$\beta_1 = \tau_1(1+\xi_1), \tag{69}$$

$$\alpha_2 = 0.5k_2\sqrt{1+\zeta_2}, \tag{70}$$

$$\beta_2 = \tau_2(1+\xi_2), \tag{71}$$

where

$$\zeta_1 = \frac{2|WA_x^- - MA^-|^2}{W^2(1+A_3)^2\mathbf{A}_x^2} - 1, \tag{72}$$

$$\zeta_2 = \frac{2|W[(1+A_3)(1+B_3)^{-1}B^-]_x - M[(1+A_3)(1+B_3)^{-1}B^-]|^2}{W^2(1+A_3)^2\mathbf{B}_x^2} - 1, \tag{73}$$

$$\xi_1 = \frac{\bar{R}_x R - \bar{R}R_x - 4i|R|^2\nu_x}{2i\alpha_1^2 W^2(1+A_3)^2\tau_1} - 1, \tag{74}$$

$$\xi_2 = \frac{\bar{Z}_x Z - \bar{Z}Z_x - 4i|Z|^2\nu_x}{2i\alpha_2^2 W^2(1+A_3)^2\tau_2} - 1. \tag{75}$$

Here

$$W = 2 + \frac{(1+A_3)(1-B_3)}{1+B_3} = 2 + K, \qquad (76)$$

$$M = A_{3x} + \frac{A^+A_x^-}{1+A_3} + \frac{A_{3x}(1-B_3)}{1+B_3} + \frac{(1+A_3)B^+B_x^-}{(1+B_3)^2} - \frac{(1+A_3)(1-B_3)B_{3x}}{(1+B_3)^2}, \qquad (77)$$

$$R = WA_x^- - MA^-, \qquad (78)$$

$$Z = W[(1+A_3)(1+B_3)^{-1}B^-]_x - M[(1+A_3)(1+B_3)^{-1}B^-]. \qquad (79)$$

$$\nu = \partial_x^{-1}\left[\frac{A_1A_{2x} - A_{1x}A_2}{(1+A_3)W} - \frac{(1+A_3)(B_{1x}B_2 - B_1B_{2x})}{(1+B_3)^2W}\right] \qquad (80)$$

We now ready to write the equations for the functions α_i and β_j. They satisfy the following four equations

$$\alpha_{1t} - 2\alpha_{1x}\beta_1 - \alpha_1\beta_{1x} = 0, \tag{81}$$

$$\beta_{1t} + \left[\frac{\alpha_{1xx}}{\alpha_1} - \beta_1^2 + 2(\alpha_1^2 + \alpha_2^2)\right]_x = 0, \tag{82}$$

$$\alpha_{2t} - 2\alpha_{2x}\beta_2 - \alpha_2\beta_{2x} = 0, \tag{83}$$

$$\beta_{2t} + \left[\frac{\alpha_{2xx}}{\alpha_2} - \beta_2^2 + 2(\alpha_1^2 + \alpha_2^2)\right]_x = 0. \tag{84}$$

Let us now we introduce new two complex functions as

$$q_1 = \alpha_1 e^{-i\partial_x^{-1}\beta_1}, \quad (85)$$

$$q_2 = \alpha_2 e^{-i\partial_x^{-1}\beta_2}. \quad (86)$$

Sometime we use the following explicit form of the transformation (85) and (86)

$$q_1 = 0.5k_1\sqrt{1+\zeta_1}e^{-i\partial_x^{-1}[\tau_1(1+\xi_1)]}, \quad (87)$$

$$q_2 = 0.5k_2\sqrt{1+\zeta_2}e^{-i\partial_x^{-1}[\tau_2(1+\xi_2)]}. \quad (88)$$

It is not difficult to verify that these functions satisfy the following Manakov system

$$iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 = 0, \quad (89)$$

$$iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 = 0. \quad (90)$$

The vector nonlinear Schrödinger equation is associated with symmetric space $SU(n+1)/S(U(1) \otimes U(n))$ [8]. The special case $n = 2$ of such symmetric space is associated with the famous Manakov system.

[8] N.A. Kostov, R. Dandoloff, V.S. Gerdjikov and G.G. Grahovski. *The Manakov system as two moving interacting curves*, arXiv:0707.0575v1 [nlin.SI] 4 Jul 2007.

Conclusion

Thus we have shown that the Manakov system (89)-(90) is the geometrical equivalent counterpart of the 2-layer spin systems or, in other terminology, the coupled spin systems (32)-(33). It is interesting to understand the role of the constant magnetic field **H**. It seems that this constant magnetic vector plays an important role in our construction of integrable multilayer spin systems and in nonlinear dynamics of multilayer magnetic systems.

Thank you for attention!