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Perturbed $(2n-1)$-dimensional Kepler problem and the nilpotent adjoint orbits of $U(n,n)$

Anatol Odzijewicz
Institute of Mathematics
University in Białystok

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I Preliminary definitions

$(\mathbb{C}^{2n}, \phi) = \mathcal{T}$ - twistor space, where ϕ is a hermitian form on $\mathbb{C}^{2n}$ of signature $(\underbrace{+\dots+}_{n}\underbrace{-\dots-}_{n})$

$$\phi = \phi^{+}, \qquad \phi^2 = id, \qquad \phi \in Mat_{2n\times 2n}(\mathbb{C})$$

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The group of symmetry of the twistor space:

$$U(n,n) := \{g \in GL(2n,\mathbb{C}) : \; g\phi g^{+} = \phi\}$$

We define the complex vector bundle

$$\mathcal{N} := \{(\mathcal{Z}, z) \in \mathbf{gl}(2n, \mathbb{C}) \times \mathbf{Gr}(n, \mathbb{C}^{2n}) : Im(\mathcal{Z}) \subset z \subset Ker(\mathcal{Z})\} \tag{1}$$

and involutions
$I : \mathbf{gl}(2n, \mathbb{C}) \to \mathbf{gl}(2n, \mathbb{C})$,
$\perp : \mathbf{Gr}(n, \mathbb{C}^{2n}) \to \mathbf{Gr}(n, \mathbb{C}^{2n})$,
$\tilde{I} : \mathcal{N} \to \mathcal{N}$
by

$$I(\mathcal{Z}) := -\phi \mathcal{Z}^{+} \phi, \tag{2}$$

$$\perp(z) := z^{\perp}, \tag{3}$$

$$\tilde{I}(\mathcal{Z}, z) := (I(\mathcal{Z}), z^{\perp}), \tag{4}$$

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\tilde{I}} & \mathcal{N} \\
\downarrow{\scriptstyle \pi_{\mathcal{N}}} & & \downarrow{\scriptstyle \pi_{\mathcal{N}}} \\
\mathbf{Gr}(n, \mathbb{C}^{2n}) & \xrightarrow{\perp} & \mathbf{Gr}(n, \mathbb{C}^{2n})
\end{array} \tag{5}
$$

By definition

$$\mathcal{Z} \in \mathbf{u}(n,n) \quad iff \quad I(\mathcal{Z}) = \mathcal{Z}$$
$$z \in \mathbf{Gr}_0(n, \mathbb{C}^{2n}) \quad iff \quad z = z^{\perp}$$
$$(\mathcal{Z}, z) \in \mathcal{N}_0 \quad iff \quad \tilde{I}(\mathcal{Z}, z) = (\mathcal{Z}, z)$$

By $\pi_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathbf{Gr}_0(n, \mathbb{C}^{2n})$ we denote the real vector bundle over the Grassmannian $\mathbf{Gr}_0(n, \mathbb{C}^{2n})$ of complex n-dimensional subspaces of $\mathbb{C}^{2n}$ isotropic with respect to

$$\langle v, w \rangle := v^{+} \phi w \tag{6}$$

Proposition

An element $\mathfrak{X} \in \mathbf{u}(n,n)$ belongs to $pr_1(\mathcal{N}_0)$ if and only if $\mathfrak{X}^2 = 0$.

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Taking the decomposition $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ we choose as ϕ the Hermitian matrix

$$\phi_d = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \tag{7}$$

where E and 0 are unit and zero $n \times n$-matrices

There is a natural diffeomorphism of manifolds $U(n) \cong \mathbf{Gr}_0(n, \mathbb{C}^{2n})$ defined in the following way

$$I_0 : U(n) \ni Z \mapsto z := \left\{ \begin{pmatrix} Z\xi \\ \xi \end{pmatrix} : \xi \in \mathbb{C}^n \right\} \in \mathbf{Gr}_0(n, \mathbb{C}^{2n}). \quad (8)$$

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For ϕ_d the block matrix elements $A, B, C, D \in Mat_{n\times n}(\mathbb{C})$ of $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$ satisfy

$$A^+A = E + C^+C, \quad D^+D = E + B^+B \text{ and } D^+C = B^+A. \quad (9)$$

From (8) one finds that $U(n,n)$ acts on $U(n)$ as follows

$$Z' = \sigma_g(Z) = (AZ + B)(CZ + D)^{-1}. \quad (10)$$

Proposition

(i) The map $\mathsf{I}_0 : T^*U(n) \cong U(n) \times iH(n) \to \mathcal{N}_0$ defined by

$$\mathsf{I}_0(Z,\rho) := \left(\begin{pmatrix} -Z\rho Z^+ & Z\rho \\ (Z\rho)^+ & \rho \end{pmatrix}, \left\{ \begin{pmatrix} Z\xi \\ \xi \end{pmatrix} : \xi \in \mathbb{C}^n \right\} \right) \in \mathcal{N}_0 \tag{11}$$

is a $U(n,n)$-equivariant (i.e. $\mathsf{I}_0 \circ \Lambda_g = \Sigma_g \circ \mathsf{I}_0$) isomorphism of the vector bundles. The action $\Sigma_g : \mathcal{N}_0 \to \mathcal{N}_0$, $g \in U(n,n)$, is a restriction to $U(n,n)$ and $\mathcal{N}_0 \subset \mathcal{N}$ of the action of the complex linear group $GL(2n,\mathbb{C})$ $\quad \Sigma_g(\mathcal{Z}, z) := (g\mathcal{Z}g^{-1}, \sigma_g(z))$.
The action $\Lambda_g : U(n) \times iH(n) \to U(n) \times iH(n)$ is defined by

$$\Lambda_g(Z,\rho) = ((AZ+B)(CZ+D)^{-1}, (CZ+D)\rho(CZ+D)^+), \tag{12}$$

where $g = \begin{pmatrix} A & B \\ C & C \end{pmatrix}$.

Proposition

(ii) The canonical one-form γ_0 on $T^*U(n) \cong U(n) \times iH(n)$ written in the coordinates $(Z, \delta) \in U(n) \times iH(n)$ assumes the form

$$\gamma_0 = iTr(\rho Z^+ dZ) \tag{13}$$

and it is invariant with respect to the action (12).

Proposition

(iii) The map $\mathbf{J}_0 : T^*U(n) \to \mathbf{u}(n,n)$ defined by

$$\mathbf{J}_0(Z,\rho) := (pr_1 \circ \mathbf{I}_0)(Z,\rho) = \begin{pmatrix} -Z\rho Z^+ & Z\rho \\ (Z\rho)^+ & \rho \end{pmatrix}$$

is the momentum map for symplectic form $d\gamma_0$, i.e. it is a $U(n,n)$-equivariant Poisson map of symplectic manifold $(T^*U(n), d\gamma_0)$ into Lie-Poisson space $(\mathbf{u}(n,n) \cong \mathbf{u}(n,n)^*, \{\cdot,\cdot\}_{L-P})$

$$\{f,g\}_{L-P}(\alpha,\delta,\beta,\beta^+) = Tr\Bigg(\alpha\left(\left[\frac{\partial f}{\partial \alpha},\frac{\partial g}{\partial \beta}\right] + \frac{\partial f}{\partial \beta}\frac{\partial g}{\partial \beta^+} - \frac{\partial g}{\partial \beta}\frac{\partial f}{\partial \beta^+}\right)$$
$$+\beta\left(\frac{\partial f}{\partial \beta^+}\frac{\partial g}{\partial \alpha} + \frac{\partial f}{\partial \delta}\frac{\partial g}{\partial \beta^+} - \frac{\partial g}{\partial \beta^+}\frac{\partial f}{\partial \alpha} - \frac{\partial g}{\partial \delta}\frac{\partial f}{\partial \beta^+}\right)$$
$$+\beta^+\left(\frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial \beta} + \frac{\partial f}{\partial \beta}\frac{\partial g}{\partial \delta} - \frac{\partial g}{\partial \alpha}\frac{\partial f}{\partial \beta} - \frac{\partial g}{\partial \beta}\frac{\partial f}{\partial \delta}\right)$$
$$+\delta\left(\left[\frac{\partial f}{\partial \delta},\frac{\partial g}{\partial \delta}\right] + \frac{\partial f}{\partial \beta^+}\frac{\partial g}{\partial \beta} - \frac{\partial g}{\partial \beta^+}\frac{\partial f}{\partial \beta}\right)\Bigg)$$

for $f,g \in C^\infty(\mathbf{u}(n,n),\mathbb{R})$.

Proposition

(i) Any $\Lambda(U(n,n))$-orbit $\mathcal{O}_{k,l}$ in $T^*U(n) = U(n) \times iH(n)$ is univocally generated from the element $(E, \rho_{k,l}) \in U(n) \times iH(n)$, where

$$\rho_{k,l} := i\ diag(\underbrace{1,\ldots,1}_{k}\underbrace{-1,\ldots,-1}_{l}\underbrace{0,\ldots,0}_{n-k-l}) \tag{14}$$

and has structure of a trivial bundle $\mathcal{O}_{k,l} \to U(n)$ over $U(n)$, i.e. $\mathcal{O}_{k,l} \cong U(n) \times \Delta_{k,l}$, where $\Delta_{k,l} := \{F\rho_{k,l}F^+ : F \in GL(n,\mathbb{C})\}$.

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(ii) The momentum map (12) gives one-to-one correspondence $\mathcal{O}_{k,l} \leftrightarrow \mathbf{J}_0(\mathcal{O}_{k,l}) = \mathcal{N}_{k,l} \subset pr_1(\mathcal{N}_0) = \{\mathfrak{X} \in \mathbf{u}(n,n) : \mathfrak{X}^2 = 0\}$ between $\Lambda(U(n,n))$-orbits in $T^*U(n)$ and $Ad(U(n,n))$-orbits in $pr_1(\mathcal{N}_0)$, where $\mathcal{N}_{kl} = \{Ad_g \mathbf{I}_0(E,\rho_{k,l}) : g \in U(n,n)\}$.

Let us define a $U(n,n)$-invariant differential one-form

$$\gamma_{+-} := i(\eta^+ d\eta - \xi^+ d\xi) \tag{15}$$

on $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. The Poisson bracket $\{f,g\}_{+-}$ corresponding to the symplectic form $d\gamma_{+-}$ is given by

$$\{f,g\}_{+-} := i\left(\frac{\partial f}{\partial \eta^+}\frac{\partial g}{\partial \eta} - \frac{\partial g}{\partial \eta^+}\frac{\partial f}{\partial \eta} - \left(\frac{\partial f}{\partial \xi^+}\frac{\partial g}{\partial \xi} - \frac{\partial g}{\partial \xi^+}\frac{\partial f}{\partial \xi}\right)\right) \tag{16}$$

and momentum map $\mathbf{J}_{+-} : \mathbb{C}^{2n} \to \mathbf{u}(n,n)$ by

$$\mathbf{J}_{+-}(\eta, \xi) := i \begin{pmatrix} -\eta\eta^{+} & \eta\xi^{+} \\ -\xi\eta^{+} & \xi\xi^{+} \end{pmatrix}, \qquad (17)$$

where $\eta, \xi \in \mathbb{C}^n$ and $f, g \in C^{\infty}(\mathbb{C}^n \oplus \mathbb{C}^n)$. One has the following identify

$$\mathbf{J}_{+-}(\eta, \xi)^2 = (\eta^{+}\eta - \xi^{+}\xi) \cdot \mathbf{J}_{+-}(\eta, \xi) \qquad (18)$$

for this momentum map.

Hence, $\mathbf{J}_{+-}$ maps the space of null-twistors $\mathcal{T}^0_{+-} := I^{-1}_{+-}(0)$, where

$$I_{+-} := \eta^+\eta - \xi^+\xi, \tag{19}$$

onto the nilpotent coadjoint orbit $\mathcal{N}_{10} = \mathbf{J}_0(\mathcal{O}_{10})$ corresponding to $k = 1$ and $l = 0$. The Hamiltonian flow $\sigma^t_{+-} : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$, $t \in \mathbb{R}$, defined by I_{+-} is given by

$$\sigma^t_{+-} \begin{pmatrix} \eta \\ \xi \end{pmatrix} := e^{it} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{20}$$

Proposition

(i) Nilpotent orbit $\mathcal{N}_{10}$ is the total space of the fibre bundle

$$\begin{array}{ccc} \mathbb{S}^{2n-1} & \longrightarrow & \mathcal{N}_{10} \\ & & \downarrow \\ & & \dot{\mathbb{C}}^n/U(1) \end{array} \tag{21}$$

over $\dot{\mathbb{C}}^n/U(1)$ with $\mathbb{S}^{2n-1}$ as a typical fibre. So, this bundle is a bundle of $(2n-1)$-dimensional spheres associated to $U(1)$-principal bundle $\dot{\mathbb{C}}^n \to \dot{\mathbb{C}}^n/U(1)$.

III $(\mathbf{C}^{2n}, \phi)$ as $U(n,n)$-Hamiltonian space

Proposition

(ii) One can also consider $\mathcal{N}_{10}$ as the total space of the fibre bundle

$$\begin{array}{ccc} \dot{\mathbb{C}}^n & \longrightarrow & \mathcal{N}_{10} \\ & & \downarrow \\ & & \mathbb{CP}(n-1) \end{array} \tag{22}$$

over complex projective space $\mathbb{CP}(n-1)$ which is the base of Hopf $U(1)$-principal bundle $\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}/U(1) \cong \mathbb{CP}(n-1)$.

III $(\mathbf{C}^{2n}, \phi)$ as $U(n,n)$-Hamiltonian space

We also will use the anti-diagonal

$$\phi_a := i \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \qquad (23)$$

realization of twistor form (6).
Subsequently we will denote the realizations $(\mathbb{C}^{2n}, \phi_d)$ and $(\mathbb{C}^{2n}, \phi_a)$ of twistor space by $\mathcal{T}$ and $\tilde{\mathcal{T}}$, respectively. The same convention will be assumed for their groups of symmetry, i.e. $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n)$ if and only if $g^+ \phi_d g = \phi_d$ and $\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \in \widetilde{U(n,n)}$ if and only if $\tilde{g}^+ \phi_a \tilde{g} = \phi_a$. Hence, for $\tilde{g} \in \widetilde{U(n,n)}$ one has

$$\begin{aligned} \tilde{A}^+\tilde{C} &= \tilde{C}^+\tilde{A}, \\ \tilde{D}^+\tilde{B} &= \tilde{B}^+\tilde{D}, \\ \tilde{A}^+\tilde{D} &= E + \tilde{C}^+\tilde{B}. \end{aligned} \qquad (24)$$

The canonical one-form (15) and the momentum map (17) for $\tilde{\mathcal{T}}$ are given by

$$\tilde{\gamma}_{+-} = v^+ d\zeta - \zeta^+ dv \tag{25}$$

and by

$$\tilde{\mathbf{J}}_{+-}(v,\zeta) = \begin{pmatrix} v\zeta^+ & -vv^+ \\ \zeta\zeta^+ & -\zeta v^+ \end{pmatrix}, \tag{26}$$

where $\begin{pmatrix} v \\ \zeta \end{pmatrix} \in \tilde{\mathcal{T}}$. The null twistors space is defined as $\tilde{\mathcal{T}}^0_{+-} := \tilde{I}^{-1}_{+-}(0)$, where

$$\tilde{I}_{+-}(v,\zeta) := i(\zeta^+ v - v^+ \zeta). \tag{27}$$

The Hamiltonian flow on $\mathbb{C}^{2n}$ generated by $\tilde{I}_{+-}$ is given by

$$\tilde{\sigma}^t_{+-}\begin{pmatrix} v \\ \zeta \end{pmatrix} = e^{it}\begin{pmatrix} v \\ \zeta \end{pmatrix} \in \tilde{\mathcal{T}}. \tag{28}$$

Both realizations $\mathcal{T}$ and $\tilde{\mathcal{T}}$ of the twistor space are related by the following unitary transform of $\mathbb{C}^{2n}$:

$$\begin{pmatrix} v \\ \zeta \end{pmatrix} = \mathcal{C}^{+}\begin{pmatrix} \eta \\ \xi \end{pmatrix} \text{ and } \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \mathcal{C}\begin{pmatrix} v \\ \zeta \end{pmatrix}, \tag{29}$$

where

$$\mathcal{C} := \frac{1}{\sqrt{2}}\begin{pmatrix} E & -iE \\ -iE & E \end{pmatrix}. \tag{30}$$

Now let us consider $H(n) \times H(n)$ with $d\tilde{\gamma}_0$, where

$$\tilde{\gamma}_0 := -Tr(XdY) \tag{31}$$

and $(Y, X) \in H(n) \times H(n)$, as a symplectic manifold. We define the symplectic action of $\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ on $H(n) \times H(n)$ by

$$\tilde{\sigma}_{\tilde{g}}(Y, X) := ((\tilde{A}Y + \tilde{B})(\tilde{C}Y + \tilde{D})^{-1}, (\tilde{C}Y + \tilde{D})X(\tilde{C}Y + \tilde{D})^{+}). \tag{32}$$

We note that this action is not defined globally, i.e. the formula (32) is valid only if $\det(\tilde{C}Y + \tilde{D}) \neq 0$.

The momentum map $\tilde{\mathbf{J}}_0 : H(n) \times H(n) \to \widetilde{\mathbf{u}(n, n)}$ corresponding to $d\tilde{\gamma}_0$ and $\tilde{\sigma}_{\tilde{g}}$ has the form

$$\tilde{\mathbf{J}}_0(Y, X) = \begin{pmatrix} -YX & YXY \\ -X & XY \end{pmatrix} \tag{33}$$

and it satisfies the equivariance property $\tilde{\mathbf{J}} \circ \tilde{\sigma}_{\tilde{g}} = Ad_{\tilde{g}} \circ \tilde{\mathbf{J}}$.

Proposition

All arrows in the above diagram are the $U(n,n)$-equivariant Poisson maps:

$$\begin{array}{ccccc} T^*U(n) & \xrightarrow{\quad \mathbf{J}_0 \quad} & \mathbf{u}(n,n) & \xleftarrow{\quad \mathbf{J}_{+-} \quad} & \mathcal{T}, \\ \uparrow T^*_{\mathcal{C}} & & \uparrow Ad_{\mathcal{C}} & & \uparrow \mathcal{C} \\ H(n)\times H(n) & \xrightarrow{\quad \tilde{\mathbf{J}}_0 \quad} & \widetilde{\mathbf{u}(n,n)} & \xleftarrow{\quad \tilde{\mathbf{J}}_{+-} \quad} & \tilde{\mathcal{T}} \end{array} \qquad (34)$$

where by the definition one has $Ad_{\mathcal{C}}(\tilde{\mathfrak{X}}) := \mathcal{C}\mathfrak{X}\mathcal{C}^+$,

$$T^*_{\mathcal{C}}(Y,X) = ((Y-iE)(-iY+E)^{-1}, \frac{i}{2}(-iY+E)X(-iY+E)^+),$$

for $\tilde{\mathfrak{X}} \in \widetilde{\mathbf{u}(n,n)}$ and $(X,Y) \in H(n)\times H(n)$.

The component

$$Z = (Y - iE)(-iY + E)^{-1}$$

is a smooth one-to-one map of $H(n)$ into $U(n)$, which is known as Cayley transform. Hence, the unitary group $U(n)$ could be considered as a compactification of $H(n)$, Namely, in order to obtain the full group $U(n)$ one adds to Cayles image of $H(n)$ such unitary matrices Z, which satisfy the condition $\det(iZ + E) = 0$. Thus the inverse Cayley map is defined by

$$Y = (Z + iE)(iZ + E)^{-1}, \quad (35)$$

if $\det(iZ + E) \neq 0$.

IV Equivalent realization of the regularized Kepler problem

We complete the above commutative diagram by the following $U(n,n)$-equivariant maps

$$\begin{array}{ccc} U(n)\times\dot{\mathbb{C}}^n & \overset{\iota}{\hookrightarrow} & T^*U(n) \\ \uparrow S_{\mathcal{C}} & & \uparrow T^*_{\mathcal{C}} \\ H(n)\times\dot{\mathbb{C}}^n & \overset{\tilde{\iota}}{\hookrightarrow} & H(n)\times H(n), \end{array} \tag{36}$$

where

$$S_{\mathcal{C}}(Y,\zeta) := ((Y-iE)(-iY+E)^{-1}, \frac{1}{\sqrt{2}}(-iY+E)\zeta),$$

$$\iota(Z,\xi) := (Z, i\xi\xi^+),$$

$$\tilde{\iota}(Y,\zeta) := (Y, \zeta\zeta^+).$$

IV Equivalent realization of the regularized Kepler problem

The following statements are valid:

$$\iota(U(n)\times\dot{\mathbb{C}}^n)=\mathcal{O}_{10},\qquad \mathbf{J}_0(\mathcal{O}_{10})=\mathcal{N}_{10}=\mathbf{J}_{+-}(\mathcal{T}^0_{+-}),$$

$$\dot{\tilde{\mathcal{O}}}_{10}:=\tilde{\iota}(H(n)\times\dot{\mathbb{C}}^n)=\{(Y,X):\dim(Im(X))=1 \text{ and } X\geq 0\},$$

$$\tilde{\mathbf{J}}_0(\dot{\tilde{\mathcal{O}}}_{10})\subset\tilde{\mathcal{N}}_{10}=\tilde{\mathbf{J}}_{+-}(\tilde{\mathcal{T}}^0_{+-}).$$

From above equalities one finds the morphism of symplectic manifolds

$$\begin{array}{ccccc}
\mathcal{O}_{10}/_\sim & \xrightarrow{\mathbf{J}_0/_\sim} & \mathcal{N}_{10} & \xleftarrow{\mathbf{J}_{+-}/_\sim} & \mathcal{T}^0_{+-}/_\sim \\
\uparrow{\scriptstyle T^*_{\mathcal{C}}/_\sim} & & \uparrow{\scriptstyle Ad_{\mathcal{C}}/_\sim} & & \uparrow{\scriptstyle \mathcal{C}/_\sim} \\
\dot{\tilde{\mathcal{O}}}_{10}/_\sim & \xhookrightarrow{\tilde{\mathbf{J}}_0/_\sim} & \tilde{\mathcal{N}}_{10} & \xleftarrow{\tilde{\mathbf{J}}_{+-}/_\sim} & \tilde{\mathcal{T}}^0_{+-}/_\sim,
\end{array}\qquad (37)$$

which are symplectic isomorphisms (except of $T^*_{\mathcal{C}}/_\sim:\dot{\tilde{\mathcal{O}}}_{10}/_\sim\hookrightarrow\mathcal{O}_{10}/_\sim$ and $\tilde{\mathbf{J}}_0/_\sim:\dot{\tilde{\mathcal{O}}}_{10}/_\sim\hookrightarrow\tilde{\mathcal{N}}_{10}$). The equivalence relations $\sim$ are defined by the reductions of respective symplectic structures from the previous diagram.

IV Equivalent realization of the regularized Kepler problem

Any element $\mathfrak{X} = \begin{pmatrix} a & b \\ b^+ & d \end{pmatrix} \in \mathbf{u}(n,n)$ defines the linear function

$$L_{\mathfrak{X}} \begin{pmatrix} \alpha & \beta \\ \beta^+ & \delta \end{pmatrix} := Tr\left(\begin{pmatrix} a & b \\ b^+ & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^+ & \delta \end{pmatrix} \right) \tag{38}$$

on the Lie-Poisson space $(\mathbf{u}(n,n), \{\cdot,\cdot\}_{L-P})$, where the Lie-Poisson bracket $\{\cdot,\cdot\}_{L-P}$ is defined in (12). These functions satisfy

$$\{L_{\mathfrak{X}_1}, L_{\mathfrak{X}_2}\}_{L-P} = L_{[\mathfrak{X}_1,\mathfrak{X}_2]}. \tag{39}$$

In the case $\mathfrak{X}_{++} = i \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$ and $\mathfrak{X}_{+-} = i \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$ one obtains

$$\left(L_{\mathfrak{X}_{++}} \circ \mathbf{J}_{+-}\right)(\eta,\xi) = \eta^+\eta - \xi^+\xi = I_{+-}, \tag{40}$$

$$\left(L_{\mathfrak{X}_{++}} \circ \mathbf{J}_{0}\right)(Z,\rho) = 0, \tag{41}$$

$$\left(L_{\mathfrak{X}_{+-}} \circ \mathbf{J}_{+-}\right)(\eta,\xi) = \eta^+\eta + \xi^+\xi =: I_{++}, \tag{42}$$

$$\left(L_{\mathfrak{X}_{+-}} \circ \mathbf{J}_{0}\right)(Z,\rho) = -2iTr\rho =: I_0. \tag{43}$$

Rewriting the above formula in the anti-diagonal realization, where $\tilde{\mathfrak{X}}_{++} = i\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = \mathcal{C}\mathfrak{X}_{++}\mathcal{C}^{+}$ and $\tilde{\mathfrak{X}}_{+-} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} = \mathcal{C}\mathfrak{X}_{+-}\mathcal{C}^{+}$ we find

$$\left(L_{\tilde{\mathfrak{X}}_{++}} \circ \tilde{\mathbf{J}}_{+-}\right)(v,\zeta) = i(v\zeta^{+} - \zeta v^{+}), \tag{44}$$

$$\left(L_{\tilde{\mathfrak{X}}_{++}} \circ \tilde{\mathbf{J}}_{0}\right)(Y,X) = 0, \tag{45}$$

$$\left(L_{\tilde{\mathfrak{X}}_{+-}} \circ \tilde{\mathbf{J}}_{+-}\right)(v,\zeta) = v^{+}v + \zeta^{+}\zeta =: \tilde{I}_{++}, \tag{46}$$

$$\left(L_{\tilde{\mathfrak{X}}_{+-}} \circ \tilde{\mathbf{J}}_{0}\right)(Y,X) = Tr(X(E+Y^{2})) =: \tilde{I}_{0}. \tag{47}$$

IV Equivalent realization of the regularized Kepler problem

The functions $I_{++}, I_0, \tilde{I}_{++}$ and $\tilde{I}_0$ are invariants of the Hamiltonian flows generated by $i\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \in \mathbf{u}(n,n)$. So, they could be considered as Hamiltonians (generators of Hamiltonian flows) on the reduced symplectic manifolds $\mathcal{T}^0_{+-}/_\sim$, $\mathcal{O}_{10}/_\sim$, $\tilde{\mathcal{T}}^0_{+-}/_\sim$ and $\dot{\tilde{\mathcal{O}}}_{10}/_\sim$, respectively. Taking into account the symplectic manifolds isomorphisms mentioned in the diagram (37) and the commutativity of the Poisson maps from (34), we conclude that:

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(i) the Hamiltonian systems: $(\mathcal{T}^0_{+-}/_\sim, I_{++})$, $(\tilde{\mathcal{T}}^0_{+-}/_\sim, \tilde{I}_{++})$, $(\mathcal{O}_{10}/_\sim, I_0)$ are isomorphic with the Hamiltonian system $(\mathcal{N}_{10}, L_{\mathfrak{X}_{+-}})$;

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(ii) the Hamiltonian system $(\dot{\tilde{\mathcal{O}}}_{10}/_\sim, \tilde{I}_0)$ is extended (regularized) by symplectic map $(T^*_{\mathcal{C}}/_\sim) : \dot{\tilde{\mathcal{O}}}_{10}/_\sim \hookrightarrow \mathcal{O}_{10}/_\sim$ to the Hamiltonian system $(\mathcal{O}_{10}/_\sim, I_0)$.

IV Equivalent realization of the regularized Kepler problem

Integrals of motion $M : H(n) \times H(n) \to H(n)$ and $R : H(n) \times H(n) \to H(n)$ for the Hamiltonian $\tilde{I}_0$ are given in matrix form by

$$M := i[X, Y] \quad \text{and} \quad R := X + YXY. \tag{48}$$

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The Hamilton equations defined by $\tilde{I}_0$ are

$$\begin{aligned} \frac{d}{dt}Y &= E + Y^2, \\ \frac{d}{dt}X &= -(XY - YX), \end{aligned} \tag{49}$$

i.e. they could be classified as a matrix Riccati type equations. In order to obtain the solution of (49) we note that after passing to $(\mathcal{T}^0_{+-}/_\sim, I_{++})$ they asssume the form of a linear equations solved by

$$\sigma^t_{+-}\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} e^{it}E & 0 \\ 0 & e^{-itE} \end{pmatrix}\begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{50}$$

i.e. the Hamiltonian flow is generated by $\mathfrak{X}_{+-} \in \mathbf{u}(n, n)$.

Therefore, going through the symplectic manifold isomorphism presented in (37), we obtain the solution

$$\begin{aligned} Y(t) &= (Y\cosh t - iE\sinh t)(iY\sinh t + E\cosh t)^{-1} \\ X(t) &= (iY\sinh t + E\cosh t)X(iY\sinh t + E\cosh t)^{+} \end{aligned} \tag{51}$$

of (32), by specifying the transformation formula (29) to
$g(t) = \mathcal{C}^{+}\begin{pmatrix} e^{t}E & 0 \\ 0 & e^{-tE} \end{pmatrix}\mathcal{C}$.

V Generalization of Kuastaanheimo-Stiefel transformation

We consider the case $n = 2$ in details. Using the Poisson morphism presented in the lower lines of (34) and (37) we find the following relations

$$X = \zeta\zeta^+, \tag{52}$$

$$v = Y\zeta \tag{53}$$

between $(Y, X) \in H(n) \times H(n)$ and $\begin{pmatrix} v \\ \zeta \end{pmatrix} \in \tilde{\mathcal{T}}^0_{+-}$. The equation (52) is equivalent to the conditions $\det X = 0$ and $0 \neq X \geq 0$.

For fixed $\begin{pmatrix} \upsilon \\ \zeta \end{pmatrix} \in \tilde{\mathcal{T}}^0_{+-}$ the two solutions Y_1 and Y_2 of the equation (53) are related by

$$Y_2 = Y_1 + t\varepsilon\bar{\zeta}(\varepsilon\bar{\zeta})^+, \tag{54}$$

where $\varepsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\zeta \in \dot{\mathbb{C}}^n$ and $t \in \mathbb{R}$. Expanding $(Y, X) \in H(2) \times H(2)$ in Pauli matrices $\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, i.e.

$$Y = y_\mu \sigma_\mu \text{ and } X = x^\mu \sigma_\mu, \tag{55}$$

we find that

$$\tilde{\gamma}_0 = 2y_\mu dx^\mu. \tag{56}$$

The elements $\varepsilon\bar{\zeta}(\varepsilon\bar{\zeta})^{+}$, where $\zeta \in \dot{\mathbb{C}}^{n}$, spans the degeneracy direction for symplectic form $d\tilde{\gamma}_0$ restricted to submanifold of $H(2) \times H(2)$ defined by the considtions $\det X = 0$ and $0 \neq X \geq 0$. Therefore, assuming in (54) $t = -\frac{1}{\zeta^{+}\zeta} Tr(Y_1)$, we find that equation (53) has unique solution $Y \in H(2)$ such that $2y_0 = Tr(Y) = 0$. From $y_0 = 0$ and $\det X = x^0 - \vec{x}^2 = 0$ we see that $(\vec{y}, \vec{x}) \in \mathbb{R}^3 \times \dot{\mathbb{R}}^3$ can be considered as a canonical coordinates on the reduced phase space

$$\mathcal{P}_0 := \{(Y, X) \in H(2) \times \dot{H}(2) : Tr(Y) = 0 \text{ and } \det X = 0, 0 \neq X \geq 0\}, \tag{57}$$

where $\dot{H}(2) := H(2) \backslash \{0\}$ and $\dot{\mathbb{R}}^3 := \mathbb{R}^3 \backslash \{0\}$. The above means that $\mathcal{P}_0 \cong \mathbb{R}^3 \times \dot{\mathbb{R}}^3$ and the canonical form $\tilde{\gamma}_0$ after restriction to $\mathcal{P}_0$ is given by

$$\tilde{\gamma}_0|_{\mathcal{P}_0} = 2\vec{y}d \cdot \vec{x} = 2y_k dx^k. \tag{58}$$

Using the identity

$$\sigma_k \sigma_l + \sigma_l \sigma_k = 2\delta_{kl} \tag{59}$$

for Pauli matrices σ_k, $k = 1, 2, 3$, we find that the Hamiltonian $\tilde{I}_0$, defined in (47), after restriction to $\mathcal{P}_0$ assumes the following form

$$H_0 = \tilde{I}_0|_{\mathcal{P}_0} = \|\vec{x}\| \left(1 + \vec{y}^2\right). \tag{60}$$

Let us note that $\|\vec{x}\| = x_0 = \zeta^+\zeta > 0$.

Summing up the above facts we state that the Hamiltonian system $(H(2) \times H(2), d\tilde{\gamma}_0, \tilde{I}_0)$ after reduction to $(\mathcal{P}_0, 2d\vec{y} \wedge d\vec{x}, H_0)$ is exactly the 3-dimensional Kepler system written in the "fictitious time"s which is related to the real time t via the rescaling

$$\frac{ds}{dt} = \frac{1}{\|\vec{x}\|}. \tag{61}$$

In order to express $(\vec{y}, \vec{x}) \in \mathbb{R}^3 \times \dot{\mathbb{R}}^3$ by $\begin{pmatrix} v \\ \zeta \end{pmatrix} \in \tilde{\mathcal{T}}^0_{+-}$ we put $Y = \vec{y}\vec{\sigma} = y_k \sigma_k$ and multiply the equation (53) by $\zeta^+ \sigma_l$. Then, using (59) and (52) we obtain the one-to-one map

$$\vec{y} = \frac{1}{\zeta^+ \zeta} \frac{1}{2} (v^+ \vec{\sigma} \zeta + \zeta^+ \vec{\sigma} v), \quad \vec{x} = \zeta^+ \vec{\sigma} \zeta, \tag{62}$$

of $\dot{\tilde{\mathcal{T}}}^0_{+-}/_\sim$ onto $\mathcal{P}_0$, where $\dot{\tilde{\mathcal{T}}}^0_{+-} := \tilde{\mathcal{T}}^0_{+-} \backslash \{(v, 0)^T \in \mathbb{C}^2 \oplus \mathbb{C}^2 : v \in \mathbb{C}^2\}$. This map is known in literature of celestial mechanics as Kuastaanheimo-Stiefel transformation.

Here this transformation $\kappa : \dot{\tilde{\mathcal{T}}}^0_{+-}/_\sim \to \mathcal{P}_0$ is a restriction

$$\begin{array}{ccc} \dot{\tilde{\mathcal{T}}}^0_{+-}/_\sim & \xrightarrow{\quad\kappa\quad} & \mathcal{P}_0 \cong \tilde{\mathcal{O}}_{10}/_\sim \\ \big\downarrow & & \big\downarrow T^*_{\mathcal{C}}/_\sim \\ \tilde{\mathcal{T}}^0_{+-}/_\sim & \xrightarrow{\quad\tilde{\kappa}\quad} & \mathcal{O}_{10}/_\sim \end{array} \tag{63}$$

to $\dot{\tilde{\mathcal{T}}}^0_{+-}$ of the symplectic diffeomorphism $\tilde{\kappa} : \tilde{\mathcal{T}}^0_{+-} \stackrel{\sim}{\to} \mathcal{O}_{10}/_\sim$, defined as the superposition of respective symplectic diffeomorphisms from diagram (37).

V Generalization of Kuastaanheimo-Stiefel transformation

• It is reasonable to interpret Hamiltonian systems $(\mathcal{T}^0_{+-}/_\sim, I_{++})$, $(\tilde{\mathcal{T}}^0_{+-}/_\sim, \tilde{I}_{++})$, $\mathcal{O}_{10}/_\sim, I_0)$ and $(\mathcal{N}_{10}, L_{\mathcal{X}_{+-}})$ as the various equivalent realizations of the **regularized** $(2n-1)$-**dimensional Kepler system**.

• In the particular case the symplectic diffeomorphism $\tilde{\kappa} : \tilde{\mathcal{T}}^0_{+-} \stackrel{\sim}{\to} \mathcal{O}_{10}/_\sim$ could be considered as a **generalization of Kuastaanheimo-Stiefel transformation** for the arbitrary dimension.

VI Generalized $(2n-1)$-dimensional Kepler problem

Assuming for $z \in \mathbb{C}$ and $l \in \mathbb{Z}$ the convention

$$z^l := \begin{cases} z^l & \text{for } l \geq 0 \\ \bar{z}^{-l} & \text{for } l < 0 \end{cases} \tag{64}$$

we define the following Hamiltonian

$$\begin{aligned} H = h_0(|\eta_1|^2, \ldots, |\eta_n|^2, |\xi_1|^2, \ldots, |\xi_n|^2) \\ + g_0(|\eta_1|^2, \ldots, |\eta_n|^2, |\xi_1|^2, \ldots, |\xi_n|^2) \\ \times (\eta_1^{k_1} \ldots \eta_n^{k_n} \xi_1^{l_1} \ldots \xi_n^{l_n} + \eta_1^{-k_1} \ldots \eta_n^{-k_n} \xi_1^{-l_1} \ldots \xi_n^{-l_n}), \end{aligned} \tag{65}$$

on the symplectic manifold $(\mathbb{C}^{2n}, d\gamma_{+-})$, where h_0 and g_0 are arbitrary smooth functions of $2n$ real variables and $k_1, \ldots k_n, l_1, \ldots, l_n \in \mathbb{Z}$, where $k_1 + \ldots + k_n = 0 = l_1 + \ldots + l_n$. Since $\{I_{+-}, H\} = 0$ one can reduce this system to $\mathcal{T}^0_{+-}/\sim$.

VI Generalized $(2n-1)$-dimensional Kepler problem

Ending, we write the Hamiltonian (65) in the more explicit form for the case $n=2$, i.e on $(\mathbb{R}^3 \times \dot{\mathbb{R}}^3, 2d\vec{y} \wedge d\vec{x})$.
In this case the integrals of motion M and R can be written in terms of Pauli matrices

$$M = M_0 E + \vec{M} \cdot \vec{\sigma} \quad \text{and} \quad R = R_0 E + \vec{R} \cdot \vec{\sigma},$$

where $M_0 = 0$, $R_0 = \frac{1}{2}||\vec{x}||(1+\vec{y}^2)$ and

$$\vec{M} = 2\vec{y} \times \vec{x}$$

$$\vec{R} = (1-\vec{y}^2)\vec{x} + 2\vec{y}(\vec{x} \cdot \vec{y})$$

are angular momentum and Runge-Lenz vector, respectively.

VI Generalized $(2n-1)$-dimensional Kepler problem

Using the linear relation

$$\begin{pmatrix} |\eta_1|^2 \\ |\eta_2|^2 \\ |\xi_1|^2 \\ |\xi_2|^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} R_0 \\ R_3 \\ M_0 \\ M_3 \end{pmatrix}$$

and define $M_+ := M_1 + iM_2$ and $M_- := M_1 - iM_2$

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and define $M_+ := M_1 + iM_2$ and $M_- := M_1 - iM_2$ we write the Hamiltonian H as follows

$$\tilde{H} = \tilde{h}_0(R_0, R_3, M_0, M_3) + \tilde{g}_0(R_0, R_3, M_0, M_3)\times$$

$$\times((R_\sigma - M_\sigma)^k (R_{\sigma'} + M_{\sigma'})^l + (R_{-\sigma} - M_{-\sigma})^k (R_{-\sigma'} + M_{-\sigma'})^l),$$

where $\sigma, \sigma' = +, -$, $k, l \in \mathbb{N} \cup \{0\}$ and $\tilde{h}_0$, $\tilde{g}_0$ are arbitrary smooth functions. Let us note that $R_0 = \frac{1}{2} I_0$.

THANK YOU FOR ATTENTION