Geometry, Integrability and Quantization
Varna, 2-7.06.2018

Symmetries of the space of connections on a principal G-bundle and related symplectic structures

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joint work with Grzegorz Jakimowicz and Anatol Odzijewicz
„Symmetries of the space of connections on a principal G-bundle and related symplectic structures", arXiv:1712.08420

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• The space $ConnP(M,G)$ of connections on the principal G-bundle $P(M,G)$.

• The space $CanT^*P$ of fibre-wise linear differential one-forms γ on the cotangent bundle T^*P, which annihilate the vectors tangent to the fibres of T^*P.

- G-principal bundle over a manifold M

$$\begin{tikzcd} G \arrow[r] & P \arrow[d, "\mu"] \\ & M \cong P/G \end{tikzcd}$$

where the free action of G we denote by

$$\kappa : P \times G \to P, \quad \kappa(p, g) := pg$$

and

$$\kappa_g : P \to P \qquad \kappa_g(p) := pg$$

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where $X_g \in T_gG$, $Y_h \in T_hG$ and $L_g(h) := gh$, $R_g(h) := hg$.

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where $X_g \in T_gG$, $Y_h \in T_hG$ and $L_g(h) := gh$, $R_g(h) := hg$.
For $e \in G$ - unit element of G and $\mathbf{0} : G \to TG$ - zero section of the tangent bundle TG one has

$$X_e \bullet Y_e = X_e + Y_e, \qquad \mathbf{0}_g \bullet \mathbf{0}_h = \mathbf{0}_{gh}, \qquad (3)$$

$$X_g \bullet Y_e \bullet X_g^{-1} = (TR_{g^{-1}}(e) \circ TL_g(e))Y_e =: Ad_g Y_e \qquad (4)$$

So, the Lie algebra T_eG could be considered as an abelian normal subgroup of TG and the zero section $\mathbf{0} : G \to TG$ is a group monomorphism.

Action of TG on TP

The diffeomorphism $I : G \times T_eG \to TG$

$$I(g, X_e) = TR_g(e)X_e =: X_g \tag{5}$$

allows us to consider TG as the semidirect product $G \ltimes_{Ad_G} T_eG$ of G by the T_eG, where the group product of $(g, X_e), (h, Y_e) \in G \ltimes_{Ad_G} T_eG$ is given by

$$(g, X_e) \bullet (h, Y_e) = I^{-1}(I(g, X_e) \cdot I(h, Y_e)) = \tag{6}$$

$$= (gh, X_e + T(R_{g^{-1}} \circ L_g)(e)Y_e) = (gh, X_e + Ad_gY_e).$$

Action of TG on TP

Using the above isomorphisms we obtain the action of $G \ltimes_{Ad_G} T_eG$ on the tangent bundle TP as the tangent to κ

$$\Phi_{(g,X_e)}(v_p) := T\kappa_{g,p}((g,X_e),v_p) = T\kappa_g(p)v_p + T\kappa_p(g)TR_g(e)X_e \tag{7}$$

.

Applying the above action we obtain the following isomorphisms

$$TP/T^vP \cong TP/T_eG, \tag{8}$$

$$TP/TG \cong (TP/T_eG)/G \cong (TP/G)/T_eG, \tag{9}$$

$$TM = T(P/G) \cong TP/TG, \tag{10}$$

of vector bundles, where we write $T^vP := KerT\mu$ for the vertical subbundle of TP.

- We consider the group $Aut_0(TP)$ of smooth automorphisms

$$\begin{tikzcd} TP \arrow[r, "A"] \arrow[d, "\mu"'] & TP \arrow[d, "\mu"] \\ P \arrow[r, "id"] & P \end{tikzcd}$$

Groups of automorphisms of TP

- We consider the group $Aut_0(TP)$ of smooth automorphisms

$$\begin{tikzcd} TP \arrow[r, "A"] \arrow[d, "\mu"'] & TP \arrow[d, "\mu"] \\ P \arrow[r, "id"] & P \end{tikzcd}$$

$A(p) : T_pP \to T_pP$ is an isomorphism of the tangent space T_pP depends smoothly on p

- $Aut_0(TP)$ is a normal subgroup of the group $Aut(TP)$ of all automorphisms of TP.

• The subgroup $Aut_{TG}(TP) \subset Aut_0(TP)$ consisting of those elements of $Aut_0(TP)$ whose action on TP commutes with the action (7) of $TG \cong G \ltimes_{Ad_g} T_eG$ on TP, i.e.

$$\begin{tikzcd} TP \arrow[r, "A"] \arrow[d, "\Phi"'] & TP \arrow[d, "\Phi"] \\ TP \arrow[r, "A"] & TP \end{tikzcd}$$

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$$\begin{tikzcd} TP \arrow[r, "A"] \arrow[d, "\Phi"'] & TP \arrow[d, "\Phi"] \\ TP \arrow[r, "A"] & TP \end{tikzcd} \qquad A(pg) \circ \Phi_{(g,X_e)} = \Phi_{(g,X_e)} \circ A(p)$$

• The group $Aut_{TG}(TP)$ acts also on vector bundles $TP/G \to M$ and $TM \to M$.

• We define the subgroup $Aut_N TP \subset Aut_{TG} TP$ consisting of $A \in Aut_{TG} TP$ such that $A(p) = id_p + B(p)$, where $B(p) : T_p P \to T_p^v P$.
From the definition of $B(p)$ one has $Im B(p) \subset T_p^v P \subset Ker B(p)$.
Thus $B_1(p)B_2(p) = 0$ for any $id + B_1,\ id + B_2 \in Aut_N P$. So, one has

$$(id_p + B_1(p))(id_p + B_2(p)) = id_p + B_1(p) + B_2(p). \qquad (11)$$

This shows that $Aut_N TP$ is a commutative subgroup of $Aut_{TG} TP$.

Proposition

One has the following short exact sequence

$$\{0\} \to Aut_N TP \xrightarrow{\iota} Aut_{TG} TP \xrightarrow{\lambda} Aut_0 TM \to \{id\} \qquad (12)$$

of the group morphisms,
where ι is the inclusion map
and λ is an epimorphism covering the identity map of M defined by

$$\lambda(A)(\mu(p))(T\mu(p))v_p := (T\mu(p) \circ A(p))v_p \qquad (13)$$

for $v_p \in T_pP$.

Connection form

• A connection form on P is a T_eG-valued differential one-form α satisfying the conditions

$$\alpha_p \circ T\kappa_p(e) = id_{T_eG} \tag{14}$$

$$\alpha_{pg} \circ T\kappa_g(p) = Ad_{g^{-1}} \circ \alpha_p \tag{15}$$

valid for value α_p of α at $p \in P$ and $g \in G$.
Using α one defines the decomposition

$$T_pP = T_p^vP \oplus T_p^{\alpha,h}P \tag{16}$$

of T_pP on the vertical T_p^vP and the horizontal $T_p^{\alpha,h}P := Ker\alpha_p$ subspaces which also satisfy the G-equivariance properties

$$T\kappa_g(p)T_p^vP = T_{pg}^vP, \tag{17}$$

$$T\kappa_g(p)T_p^{\alpha,h}P = T_{pg}^{\alpha,h}P. \tag{18}$$

From the decomposition (16) for any $p \in P$ one obtains the vector spaces isomorphism

$$\Gamma_\alpha(p) : T_{\mu(p)}M \to T_p^{\alpha,h}P \tag{19}$$

such that

$$\Gamma_\alpha(pg) = T\kappa_g(p) \circ \Gamma_\alpha(p) \tag{20}$$

and

$$T\mu(p) \circ \Gamma_\alpha(p) = id_{\mu(p)}, \qquad \Gamma_\alpha(p) \circ T\mu(p) = \Pi^h_\alpha(p), \tag{21}$$

where $\Pi^h_\alpha(p)$ is defined by the decomposition

$$id_p = \Pi^v_\alpha(p) + \Pi^h_\alpha(p) \tag{22}$$

of the identity map of T_pP on the projections corresponding to (16).

Proposition

A fixed connection α defines the injection

$$\sigma_\alpha : Aut_0TM \to Aut_{TG}TP$$

$$\sigma_\alpha(\tilde{A})(p) := \Pi^v_\alpha(p) + \Gamma_\alpha(p) \circ \tilde{A}(\mu(p)) \circ T\mu(p), \qquad (23)$$

where $\tilde{A} \in Aut_0TM$, the surjection $\beta_\alpha : Aut_{TG}TP \to Aut_NTP$ by $\beta_\alpha(A) := A\sigma_\alpha(\lambda(A))^{-1}$, where $A \in Aut_{TG}TP$, which are arranged into the short exact sequence

$$\{\mathrm{id}_{TM}\} \to Aut_0TM \xrightarrow{\sigma_\alpha} Aut_{TG}TP \xrightarrow{\beta_\alpha} Aut_NTP \to \{id_{TP}\}$$

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$$\{\mathrm{id}_{TM}\} \to Aut_0 TM \underset{\lambda}{\overset{\sigma_\alpha}{\rightleftarrows}} Aut_{TG} TP \underset{\iota}{\overset{\beta_\alpha}{\rightleftarrows}} Aut_N TP \to \{id_{TP}\}$$

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$$\{\mathrm{id}_{TM}\} \rightarrow Aut_0 TM \underset{\lambda}{\overset{\sigma_\alpha}{\rightleftarrows}} Aut_{TG} TP \underset{\iota}{\overset{\beta_\alpha}{\rightleftarrows}} Aut_N TP \rightarrow \{id_{TP}\}$$

inverse to the sequence (12). The map σ_α is a monomorphism

$$\sigma_\alpha(\tilde{A}_1 \tilde{A}_2) = \sigma_\alpha(\tilde{A}_1)\sigma_\alpha(\tilde{A}_2)$$

of the groups and β_α satisfies

$$\beta_\alpha(A_1 A_2) = \beta_\alpha(A_1)\sigma_\alpha(\lambda(A_1))\beta_\alpha(A_2)\sigma_\alpha(\lambda(A_1))^{-1}.$$

Proposition

Using the decomposition

$$A(p) = (\mathrm{id}_p + B(p))\sigma_\alpha(\tilde{A})(p) \tag{24}$$

of $A \in Aut_{TG}TP$, where $\mathrm{id}_p + B(p) \in Aut_N TP$ and $\tilde{A} \in Aut_0 TM$,
one defines an isomorphism

$$Aut_{TG}TP \;\longrightarrow\; Aut_0 TM \ltimes_\alpha End_N TP,$$

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where the product of $(\tilde{A}_1, B_1), (\tilde{A}_2, B_2) \in Aut_0 TM \ltimes_\alpha End_N TP$ is given by

$$[(\tilde{A}_1, B_1) \cdot (\tilde{A}_2, B_2)](p) := \tag{25}$$

$$= (\tilde{A}_1(\mu(p))\tilde{A}_2(\mu(p)), B_1(p) + B_2(p) \circ \Gamma_\alpha(p) \circ \tilde{A}_1^{-1}(\mu(p)) \circ T\mu(p)).$$

Action on the space of connections

• Let $ConnP(M,G)$ be the space of all connections on $P(M,G)$. We define

$$\phi_A(\alpha)_p := \alpha_p \circ A(p)^{-1} \qquad (26)$$

the left action $\phi_A : ConnP(M,G) \to ConnP(M,G)$ of $Aut_{TG}TP$ on $ConnP(M,G)$, i.e. ϕ satisfies $\phi_{A_1 A_2} = \phi_{A_1} \circ \phi_{A_1}$ for $A_1, A_2 \in Aut_{TG}TP$.

The following proposition shows that one can define the group $Aut_{TG}TP$ in terms of connections space $ConnP(M,G)$.

Proposition

If $A \in Aut_0(TP)$ and $\phi_A(ConnP(M,G)) \subset ConnP(M,G)$ then $A \in Aut_{TG}(TP)$.

The standard symplectic form

• We recall that the standard symplectic form on T^*P is $\omega_0 = d\gamma_0$, where $\gamma_0 \in C^\infty T^*(T^*P)$ is the canonical one-form on T^*P defined at $\varphi \in T^*P$ by

$$\langle \gamma_{0\varphi}, \xi_\varphi \rangle := \langle \varphi, T\pi^*(\varphi)\xi_\varphi \rangle,$$

where $\pi^* : T^*P \to P$ is the projection of T^*P on the base and $\xi_\varphi \in T_\varphi(T^*P)$.

A linear vector field

• By definition a *linear vector field* on T^*P is a pair (ξ, χ) of vector fields $\xi \in C^\infty T(T^*P)$ and $\chi \in C^\infty TP$ such that

$$\begin{tikzcd} T^*P \arrow[r, "\xi"] \arrow[d, "\pi^*"'] & T(T^*P) \arrow[d, "T\pi^*"] \\ P \arrow[r, "\chi"] & TP \end{tikzcd}$$

defines a morphism of vector bundles. Note here that $T\pi^*(\varphi)\xi_\varphi = \chi_{\pi^*(\varphi)}$.

A linear vector field

• We will denote by $LinC^\infty T(T^*P)$ the Lie algebra of linear vector fields over the vector bundle $\pi^* : T^*P \to P$. The Lie bracket of (ξ_1, χ_1), $(\xi_2, \chi_2) \in LinC^\infty T(T^*P)$ is defined by

$$[(\xi_1, \chi_1), (\xi_2, \chi_2)] := ([\xi_1, \xi_2], [\chi_1, \chi_2])$$

and the vector space structure on $LinC^\infty T(T^*P)$ by

$$c_1(\xi_1, \chi_1) + c_2(\xi_2, \chi_2) := (c_1\xi_1 + c_2\xi_2, c_1\chi_1 + c_2\chi_2).$$

Let $LinC^\infty(T^*P)$ denote the vector space of smooth fibre-wise linear functions on T^*P. Spaces $LinC^\infty T(T^*P)$ and $LinC^\infty(T^*P)$ have structures of $C^\infty(P)$-modules defined by $f(\xi, \chi) := ((f \circ \pi^*)\xi, f\chi)$ and by $fl := (f \circ \pi^*)l$, respectively, where $f \in C^\infty(P)$ and $l \in LinC^\infty(T^*P)$.

Definition

• A differential one-form $\gamma \in C^\infty T^*(T^*P)$ is called a *generalized canonical form* on T^*P if:

(i) $\gamma_\varphi \neq 0$ for any $\varphi \in T^*P$,

(ii) $kerT\pi^*(\varphi) \subset ker\ \gamma_\varphi$

(iii) $\langle \gamma, \xi \rangle \in LinC^\infty(T^*P)$ for any $\xi \in LinC^\infty T(T^*P)$.

The space of generalized canonical forms on T^*P will be denoted by $CanT^*P$. Let us note here that $\gamma_0 \in CanT^*P$.

Proposition

There is

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where

$$\langle \Theta(A)_\varphi, \xi_\varphi \rangle := \langle \varphi, A(\pi^*(\varphi))T\pi^*(\varphi))\xi_\varphi \rangle, \tag{27}$$

for $\xi_\varphi \in T_\varphi(T^*P)$.

Corrolary

Fixing a connection α one obtains an embedding

$$\iota_\alpha : ConnP(M,G) \hookrightarrow Can_{TG}T^*P$$

defined as follows

$$\iota_\alpha(\alpha') := \varphi \circ T\pi^*(\varphi) + \varphi \circ T\kappa_{\pi^*(\varphi)}(e) \circ (\alpha'_{\pi^*(\varphi)} - \alpha_{\pi^*(\varphi)}) \circ T\pi^*(\varphi). \tag{28}$$

- A G-equivariant diffeomorphism

$$T^*P \xrightarrow{\quad I_\alpha \quad} \overline{P} \times T_e^*G$$

- A G-equivariant diffeomorphism

$$T^*P \xrightarrow{\quad I_\alpha \quad} \overline{P} \times T_e^*G$$

dependent on a fixed connection α

$$I_\alpha(\varphi) := (\Gamma_\alpha^*(\pi^*(\varphi))(\varphi), \pi^*(\varphi), \varphi \circ T\kappa_{\pi^*(\varphi)})),$$

where

$$\overline{P} := \{(\tilde{\varphi}, p) \in T^*M \times P : \ \tilde{\pi}^*(\tilde{\varphi}) = \mu(p)\}$$

is the total space of the principal bundle $\overline{P}(T^*M, G)$ being the pullback of the principal bundle $P(M, G)$ to T^*M by the projection $\tilde{\pi}^* : T^*M \to M$ of T^*M on the base M.

$$T^*P \xrightarrow{I_\alpha} \overline{P} \times T_e^*G$$

$$
\begin{tikzcd}
T^*P \arrow[r, shift left, "I_\alpha"] & \overline{P} \times T_e^*G \arrow[l, shift left, "I_\alpha^{-1}"]
\end{tikzcd}
$$

$$I_\alpha^{-1}(\tilde{\varphi}, p, \chi) = \tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p \tag{29}$$

is the inverse to I_α.
Because of the group $Aut_{TG}TP$ acts on TP, we also can define the natural right action of $Aut_{TG}TP$ on T^*P

$$(A^*\varphi)(\pi^*(\varphi)) := \varphi \circ A(\pi^*(\varphi))$$

for $A \in Aut_{TG}TP$,
and the action of G on T^*P

$$\Phi_g^*(\varphi)(pg) = (T\kappa_g(p)^{-1})^*\varphi.$$

Using I_α we transport above actions to $\overline{P} \times T_e^* G$:

$$\Lambda_\alpha(A)(\tilde{\varphi}, p, \chi) := (I_\alpha \circ A^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) =$$

$$= ((\tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p) \circ A(p) \circ \Gamma_\alpha(p), p, \chi) \tag{30}$$

and by

$$\Phi_g^*(\tilde{\varphi}, p, \chi) := (I_\alpha \circ \phi_g^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) = (\tilde{\varphi}, pg, Ad_{g^{-1}}^* \chi), \tag{31}$$

respectively.

Using $I_\alpha^{-1} : \overline{P} \times T_e^* G \to T^* P$ we pull the generalized canonical form $\Theta(A)$ back to $\overline{P} \times T_e^* G$, where

$$\langle \Theta(A)_\varphi, \xi_\varphi \rangle := \langle \varphi, A(\pi^*(\varphi)) T\pi^*(\varphi)) \xi_\varphi \rangle, \tag{32}$$

for $\xi_\varphi \in T_\varphi(T^* P)$.

For $A = (\mathrm{id}_{TP} + B)\sigma_\alpha(\tilde{A})$ we have

$$(I_\alpha^{-1})^* \Theta(A)(\tilde{\varphi}, p, \chi) = \tag{33}$$

$$= \tilde{\varphi} \circ \tilde{A}(\mu(p)) \circ T(\tilde{\pi}^* \circ pr_1)(\tilde{\varphi}, p, \chi) + \chi \circ \alpha_p \circ A(p) \circ Tpr_2(\tilde{\varphi}, p, \chi) =$$

$$= pr_1^*(\tilde{\Theta}(\tilde{A})(\tilde{\varphi}, p, \chi) + \langle pr_3(\tilde{\varphi}, p, \chi), pr_2^*(\Phi_{A^{-1}}(\alpha))(\tilde{\varphi}, p, \chi) \rangle,$$

where $pr_3(\tilde{\varphi}, p, \chi) := \chi$.

The symplectic form corresponding to (33) is given by

$$d((I_\alpha^{-1})^*\Theta(A)) = \tag{34}$$

$$= pr_1^*(d\tilde{\Theta}(\tilde{A})) + \langle d\ pr_3 \overset{\wedge}{,}\ pr_2^*(\Phi_{A^{-1}}(\alpha))\rangle + \langle pr_3, pr_2^*(d\Phi_{A^{-1}}(\alpha))\rangle.$$

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THANK YOU FOR ATTENTION