Symmetry Groups of Systems of Entangled Particles

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Quantum Entanglement

A plethora of extraordinary phenomena emerges in quantum mechanics, the quintessential being **entanglement**.

In quantum physics, entangled particles remain connected so that actions performed on one affect the other, even when separated by great distances.

Quantum entanglement is a physical phenomenon that occurs when groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others, **even when the particles are separated by a large distance**. Instead, a quantum state must be described for a system of particles as a whole.

Lorentz (Galilei) Symmetry

A physical system has Lorentz (resp. Galilei) symmetry if the relevant laws of physics are invariant under Lorentz (Galilei) transformations. Lorentz (Galilei) symmetry is one of the cornerstones of modern (classical) physics.

However, it is known that entangled particles involve Lorentz symmetry violation.

Indeed, several explorers exploit entangled particles to observe Lorentz symmetry violation.

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Accordingly, the aim of this talk is to uncover
generalized Lorentz (Galilei) transformations
that lead to the missing
symmetry groups of systems of entangled particles.

Generalized Galilei transformations are intuitively clear, but involve no entanglement.

In contrast,
Generalized Lorentz transformations are counterintuitive and involve entanglement.

The derivation of the generalized Lorentz transformations is surprisingly natural and elegant, involving elegant manipulations of real rectangular matrices of order $n \times m$ for any $m, n \in \mathbb{N}$.

The Galilei Boost of Signature $(m, n) = (1, 3)$

The application of the Galilei bi-boost $B_\infty(V)$ of signature $(m, n) = (1, 3)$ to a $(1, 3)$-particle $(t, \mathbf{x})$ in $m + n = 1 + 3$ time-space dimensions yields

$$\begin{aligned} \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} := B_\infty(V) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} t \\ v_1 t + x_1 \\ v_2 t + x_2 \\ v_3 t + x_3 \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} , \end{aligned} \tag{1}$$

Time is invariant under Galilei boosts: $t' = t$.
In contrast, $t^2 - \mathbf{x}^2/c^2$ is invariant under Lorentz boosts.

 The Galilei Bi-boost of Signature $(m, n) = (2, 3)$

is applied **collectively** to a system of 2 3-dimensional particles.

$$\begin{pmatrix} t_1' & 0 \\ 0 & t_2' \\ \mathbf{x}_1' & \mathbf{x}_2' \end{pmatrix} := B_\infty(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ v_{11} & v_{12} & 1 & 0 & 0 \\ v_{21} & v_{22} & 0 & 1 & 0 \\ v_{31} & v_{32} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}$$

$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ v_{11} t_1 + x_{11} & v_{12} t_2 + x_{12} \\ v_{21} t_1 + x_{21} & v_{22} t_2 + x_{22} \\ v_{31} t_1 + x_{31} & v_{32} t_2 + x_{32} \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix} \tag{2}$$

The generalization to the Galilei Boost of any signature (m, n), $m, n \in \mathbb{N}$, is now obvious.

Page 6 The Galilei Bi-boost: from Signature $(m, n) = (2, 3)$ to Signature (m, n), for all $m, n \in \mathbb{N}$

$$\begin{pmatrix} T' \\ X' \end{pmatrix} = B_\infty(V) \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} \begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} T \\ X + VT \end{pmatrix} \quad (3)$$

where if $(m, n)=(2, 3)$, then

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \quad X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{1,1} & x_{1,2} \\ x_{1,1} & x_{1,2} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \quad (4)$$

$$V = \begin{pmatrix} v_{1,1} & x_{1,2} \\ v_{1,1} & v_{1,2} \\ v_{1,1} & v_{1,2} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}, \quad B_\infty(V) \in \mathbb{R}^{(m+n)\times(m+n)} \quad (5)$$

We note that the Galilei bi-boost keeps the time invariant, $T'=T$.

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Corresponding to the Galilei bi-boost,

$$B_\infty(V)$$

of signature (m, n), $m, n \in \mathbb{N}$, parametrized by the velocity matrix $V \in \mathbb{R}^{n\times m}$, we seek the Lorentz bi-boost,

$$B_c(V)$$

parametrized by the velocity matrix $V \in \mathbb{R}_c^{n\times m}$, which keeps invariant the squared norm of signature (m, n),

$$t_1^2 + t_2^2 + t_3^2 + \ \ldots\ + t_m^2 - \frac{1}{c^2}(x_1^2 + x_2^2 + x_3^2 + \ \ldots\ + x_n^2) \quad (6)$$

for an arbitrarily fixed constant $c > 0$.

But, what is the c-ball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$?

Definition

(Matrix Ball, Matrix Norm). *For any $m, n \in \mathbb{N}$ and $c > 0$, the c-ball $\mathbb{R}_c^{n\times m}$ of the space of all $n \times m$ real matrices is given by*

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m} :\ \forall\lambda \in \sigma(VV^t),\ \sqrt{\lambda} < c\} \\ &= \{V \in \mathbb{R}^{n\times m} :\ \forall\lambda \in \sigma(V^tV),\ \sqrt{\lambda} < c\}\,.\end{aligned} \tag{7}$$

The matrix norm $\|V\|$ of $V \in \mathbb{R}^{n\times m}$ is defined by

$$\begin{aligned}\|V\| &= max\{\sqrt{\lambda} :\ \lambda \in \sigma(VV^t)\} \\ &= max\{\sqrt{\lambda} :\ \lambda \in \sigma(V^tV)\}\,.\end{aligned} \tag{8}$$

It is clear from the Def. that

$$\mathbb{R}_c^{n\times m} = \{V \in \mathbb{R}^{n\times m} :\ \|V\| < c\} \tag{9}$$

Theorem

For any $m, n \in \mathbb{N}$ *and* $c > 0$, *let* $V \in \mathbb{R}^{n\times m}$. *Then,* $V \in \mathbb{R}_c^{n\times m}$ *if and only if the real matrix*

$$\Gamma_V^L := \sqrt{I_n - c^{-2} V V^t}^{\,-1} \in \mathbb{R}^{n\times n} \tag{10}$$

exists, and similarly, $V \in \mathbb{R}_c^{n\times m}$ *if and only if the real matrix*

$$\Gamma_V^R := \sqrt{I_m - c^{-2} V^t V}^{\,-1} \in \mathbb{R}^{m\times m} \tag{11}$$

exists.

Obviously,

$$\begin{aligned} \lim_{c\to\infty} \Gamma_V^L &= I_n \\ \lim_{c\to\infty} \Gamma_V^R &= I_m \end{aligned} \tag{12}$$

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In the special case when $m = 1$, the right gamma factor, Γ_V^R, descends to the Lorentz gamma factor, γ_V, of special relativity theory,

$$\Gamma_V^R = \frac{1}{\sqrt{1 - c^{-2}\|V\|^2}} =: \gamma_V \qquad (m = 1)\,, \tag{13}$$

$V \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$, where $\|V\|^2 = V^t V$.

Useful Matrix Identities

$$\Gamma_V^L = I_n + \frac{1}{c^2}\frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} VV^t$$

$$\Gamma_V^R = I_m + \frac{1}{c^2}\frac{(\Gamma_V^R)^2}{I_m + \Gamma_V^R} V^tV \qquad (14)$$

for all $V \in \mathbb{R}_c^{n\times m}$.

In (14) we use the convenient *matrix division notation* A/B to denote either AB^{-1} or $B^{-1}A$ when no confusion may arise, that is, when the matrices A and B satisfy $AB^{-1} = B^{-1}A$. The identities in (14) prove useful in establishing the important additive decomposition (33).

Lorentz Transformation of signature (m, n)

Theorem

A matrix $\Lambda \in \mathbb{R}^{(m+n)\times(m+n)}$, $m, n \in \mathbb{N}$, *is the matrix representation of a Lorentz transformation* $\Lambda \in \mathrm{SO_c(m,n)}$ *of signature* (m, n) *if and only if it possesses the bi-gyration polar decomposition*

$$\Lambda = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathrm{SO_c(m,n)} = \mathrm{\mathbb{R}_c^{n\times m}} \times \mathrm{SO(m)} \times \mathrm{SO(n)}\,, \tag{15}$$

parametrized by the main parameter $V \in \mathbb{R}_c^{n\times m}$, *and the two orientation parameters* $O_m \in \mathrm{SO(m)}$ *and* $O_n \in \mathrm{SO(n)}$.

Lorentz Bi-Boost of Signature (m, n)

A Lorentz transformation of signature (m, n) without bi-rotations $(O_m, O_n) \in \mathrm{SO(m)} \times \mathrm{SO(n)}$ is called a *bi-boost*. Hence, a Lorentz bi-boost $B_c(V)$ of order (m, n) is represented by the $(m+n) \times (m+n)$ matrix

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in \mathrm{SO_c(m,n)}\,, \qquad (16)$$

parametrized by $V \in \mathbb{R}_c^{n\times m}$.

In the special case hen $m = 1$, Lorentz bi-boosts of signature $(m, n) = (1, n)$ descend to the Lorentz boosts of Einstein's special theory of relativity.

In physical applications $n = 3$, but in geometry, $n \in \mathbb{N}$.

Pseudo-Euclidean Inner Product of Signature (m, n)

Let

$$\mathbf{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \in \mathbb{R}^m, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \tag{17}$$

so that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{x} \end{pmatrix} = (t_1, \ldots, t_m, x_1, \ldots, x_n)^t \in \mathbb{R}^{m,n} \tag{18}$$

is a generic point of the pseudo-Euclidean space $\mathbb{R}^{m,n}$.

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} := \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix}^t \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -c^{-2} I_n \end{pmatrix} \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{t}_1 \cdot \mathbf{t}_2 - c^{-2} \mathbf{x}_1 \cdot \mathbf{x}_2 \tag{19}$$

for all $(\mathbf{t}_1, \mathbf{x}_1)^t, (\mathbf{t}_2, \mathbf{x}_2)^t \in \mathbb{R}^{m,n}$, where $\mathbf{t}_1 \cdot \mathbf{t}_2 = \mathbf{t}_1^t \mathbf{t}_2$ and $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^t \mathbf{x}_2$ are the standard inner product in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

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Theorem

For any $m, n \in \mathbb{N}$, the bi-boost of signature (m, n),

$$B_v(V) = \begin{pmatrix} \Gamma_V^R & c^{-2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \tag{20}$$

$V \in \mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$, leaves the pseudo-Euclidean inner product (19) invariant, that is

$$B_v(V)\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot B_v(V)\begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} \tag{21}$$

for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^m$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

Page 16 The generic (m,n)-Lorentz transformation has the form

$$\Lambda = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathrm{SO_c(m,n)} \tag{22}$$

Bi-boost Product is not a bi-boost, but it is a Lorentz transformation. Hence,

$$\begin{aligned} &\begin{pmatrix} \Gamma_{V_1}^R & \frac{1}{c^2}\Gamma_{V_1}^R V_1^t \\ \Gamma_{V_1}^L V_1 & \Gamma_{V_1}^L \end{pmatrix} \begin{pmatrix} \Gamma_{V_2}^R & \frac{1}{c^2}\Gamma_{V_2}^R V_2^t \\ \Gamma_{V_2}^L V_2 & \Gamma_{V_2}^L \end{pmatrix} \\ &= \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \end{aligned} \tag{23}$$

Einstein Velocity Addition of Signature (m, n)

$$\begin{pmatrix} \Gamma^R_{V_1} & \frac{1}{c^2}\Gamma^R_{V_1} V_1^t \\ \Gamma^L_{V_1} V_1 & \Gamma^L_{V_1} \end{pmatrix} \begin{pmatrix} \Gamma^R_{V_2} & \frac{1}{c^2}\Gamma^R_{V_2} V_2^t \\ \Gamma^L_{V_2} V_2 & \Gamma^L_{V_2} \end{pmatrix} = \begin{pmatrix} \Gamma^R_{V} & \frac{1}{c^2}\Gamma^R_{V} V^t \\ \Gamma^L_{V} V & \Gamma^L_{V} \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} \Gamma^R_{V_1} & \frac{1}{c^2}\Gamma^R_{V_1} V_1^t \\ \Gamma^L_{V_1} V_1 & \Gamma^L_{V_1} \end{pmatrix} \begin{pmatrix} \Gamma^R_{V_2} & \frac{1}{c^2}\Gamma^R_{V_2} V_2^t \\ \Gamma^L_{V_2} V_2 & \Gamma^L_{V_2} \end{pmatrix} = \begin{pmatrix} \Gamma^R_{V_1 \oplus V_2} & \frac{1}{c^2}\Gamma^R_{V_1 \oplus V_2}(V_1 \oplus V_2)^t \\ \Gamma^L_{V_1 \oplus V_2} V_1 \oplus V_2 & \Gamma^L_{V_1 \oplus V_2} \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{25}$$

Page 18 Left and Righr Gyrations:

For all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$,

$$\begin{pmatrix} \Gamma_{V_1}^R & \frac{1}{c^2}\Gamma_{V_1}^R V_1^t \\ \Gamma_{V_1}^L V_1 & \Gamma_{V_1}^L \end{pmatrix} \begin{pmatrix} \Gamma_{V_2}^R & \frac{1}{c^2}\Gamma_{V_2}^R V_2^t \\ \Gamma_{V_2}^L V_2 & \Gamma_{V_2}^L \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma_{V_1\oplus V_2}^R & \frac{1}{c^2}\Gamma_{V_1\oplus V_2}^R (V_1\oplus V_2)^t \\ \Gamma_{V_1\oplus V_2}^L V_1\oplus V_2 & \Gamma_{V_1\oplus V_2}^L \end{pmatrix} \tag{26}$$

$$\times \begin{pmatrix} O_m = \mathrm{rgyr}[V_1, V_2] & 0_{m,n} \\ 0_{n,m} & O_n = \mathrm{lgyr}[V_1, V_2] \end{pmatrix}$$

Right gyrations generated by V_1, V_2: $\mathrm{rgyr}[V_1, V_2] \in \mathrm{SO(m)}$

Left gyrations generated by V_1, V_2: $\mathrm{lgyr}[V_1, V_2] \in \mathrm{SO(n)}$

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If $m = 1$ then: $\mathrm{rgyr}[V_1, V_2] = 1$ is trivial; and
$\mathrm{lgyr}[V_1, V_2]$ is Thomas Precession.

> *The decomposition process [describing successive pure Lorentz transformations as a pure Lorentz transformation preceded, or followed, by a Thomas rotation] can be carried through on the product of two pure Lorentz transformations to obtain explicitly the rotation of the coordinate axes resulting from the two successive boosts [that is, the Thomas rotation]. In general, the algebra involved is quite forbidding, more than enough, usually, to discourage any actual demonstration of the rotation matrix.*
>
> *Herbert Goldstein, Classical Mechanics*

From Left and Right Gyrations to Gyrations

Left and right gyrations are automorphisms of $(\mathbb{R}_c^{n\times m}, \oplus)$. They are composed into gyrations according to the equation

$$\mathrm{gyr}[V_1, V_2]V = \mathrm{lgyr}[V_1, V_2]V\mathrm{rgyr}[V_2, V_1] \qquad (27)$$

for all $V, V_1, V_2 \in \mathbb{R}_c^{n\times m}$.
Left gyrations $\mathrm{lgyr}[V_1, V_2]$, right gyrations $\mathrm{rgyr}[V_1, V_2]$, and gyrosums $V_1 \oplus V_2$ are determined uniquely in terms of $V_1, V_2 \in \mathbb{R}_c^{n\times m}$ by (26).
The pair $(\mathbb{R}_c^{n\times m}, \oplus)$ along with gyrations forms a *bi-gyrocommutative bi-gyrogroup*. As such, it possesses the following elegant identities:

Bi-Gyrogroup of Signature (m, n), $m, n \in \mathbb{N}$

$$
\begin{aligned}
V_1 \oplus V_2 &= \mathrm{gyr}[V_1, V_2](V_2 \oplus V_1) && \text{Gyrocommutative Law} \\
V_1 \oplus (V_2 \oplus V_3) &= (V_1 \oplus V_2) \oplus \mathrm{gyr}[V_1, V_2] V_3 && \text{Left Gyroassociative Law} \\
(V_1 \oplus V_2) \oplus V_3 &= V_1 \oplus (V_2 \oplus \mathrm{gyr}[V_2, V_1] V_3) && \text{Right Gyroassociative Law} \\
\mathrm{gyr}[V_1 \oplus V_2, V_2] &= \mathrm{gyr}[V_1, V_2] && \text{Left Reduction Property} \\
\mathrm{gyr}[V_1, V_2 \oplus V_1] &= \mathrm{gyr}[V_1, V_2] && \text{Right Reduction Property} \\
\mathrm{gyr}[\ominus V_1, \ominus V_2] &= \mathrm{gyr}[V_1, V_2] && \text{Gyration Even Property} \\
(\mathrm{gyr}[V_1, V_2])^{-1} &= \mathrm{gyr}[V_2, V_1] && \text{Gyration Inversion Law}
\end{aligned}
\tag{28}
$$

for any $V_1, V_2, V_3 \in \mathbb{R}_c^{n \times m}$.

Einstein Scalar Multiplication of Signature (m, n)

In the special case when the signature is $(m, n) = (1, 3)$ the binary operation descends to Einstein's addition of relativistically admissible velocities. Hence, we call $\oplus$ *Einstein addition of signature* (m, n) *in the ball* $\mathbb{R}_c^{n\times m}$.
Moreover, Einstein addition admits a scalar multiplication $\otimes$, which is determined uniquely by the bi-boost identity

$$\begin{pmatrix} \Gamma^R_{r\otimes V} & \frac{1}{c^2}\Gamma^R_{r\otimes V}(r\otimes V)^t \\ \Gamma^L_{r\otimes V} r\otimes V & \Gamma^L_{r\otimes V} \end{pmatrix} = \begin{pmatrix} \Gamma^R_V & \frac{1}{c^2}\Gamma^R_V V^t \\ \Gamma^L_V V & \Gamma^L_V \end{pmatrix}^r \qquad (29)$$

The resulting triple $(\mathbb{R}_c^{n\times m}, \oplus, \otimes)$ is, thus, the bi-gyrovector space of signature (m, n).

Additive Decomposition of the (m, n)-Lorentz Bi-boost

The Lorentz bi-boost

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in \mathrm{SO_c(m,n)} \qquad (30)$$

$V \in \mathbb{R}_c^{n\times m} \subset \mathbb{R}^{n\times m}$, and its corresponding Galilei bi-boost of same signature,

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)} , \qquad (31)$$

$V \in \mathbb{R}^{n\times m}$, are related to each other by the following *additive decomposition*, which is the central result of this talk.

The Central Result

The *additive decomposition* of the Lorentz bi-boost into
(1) a Galilei transformation of signature (m, n), which is intuitively clear (as we will see); and
(2) a non-Galilean relativistic effects, which are directly noticeable only at very high speeds:

$$B_c(V) = B_\infty(V) + \frac{1}{c^2} \begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m + \Gamma_V^R} V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t V & \frac{(\Gamma_V^L)^2}{I_n + \Gamma_V^L} V V^t \end{pmatrix}, \qquad (32)$$

for all $V \in \mathbb{R}_c^{n\times m}$.
$V \in \mathbb{R}_c^{n\times m}$ represents m n-dimensional velocities:

$$V = \begin{pmatrix} \mathbf{v}_1, & \mathbf{v}_2, & \mathbf{v}_3, & \dots & \mathbf{v}_m \end{pmatrix}$$

The m columns of V represent the velocities of a system of m particles.
In Einstein's special relativity theory, $m = 1$ (and $n = 3$).

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Consequently,

the **collective application** of a Galilei bi-boost of signature (m, n) to the constituents of a system of m n-dimensional particles

is equivalent to

the **individual application** of a Galilei boost to each of the constituents of a system of m n-dimensional particles.

Hence, in particular, a collective application of a Galilei bi-boost to a system of m particles, $m > 1$, yields no entanglement of the spacetime coordinates of the particles.

This is, however, not the case when we replace Galilei bi-boosts of signature (m, n) by Lorentz bi-boosts of signature (m, n) when $m > 1$.

To see this, we return to our central result.

The Central Result

The *additive decomposition* of the Lorentz bi-boost into
(1) a Galilei transformation of signature (m, n), which is intuitively clear (as we have seen); and
(2) a non-Galilean relativistic effects, which are directly noticeable only at very high speeds:

$$B_c(V) = B_\infty(V) + \frac{1}{c^2}\begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R}V^tV & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L}VV^tV & \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L}VV^t \end{pmatrix}, \qquad (33)$$

for all $V \in \mathbb{R}_c^{n\times m}$.
$V \in \mathbb{R}_c^{n\times m}$ represents m n-dimensional velocities:

$$V = \begin{pmatrix} \mathbf{v}_1, & \mathbf{v}_2, & \mathbf{v}_3, & \dots & \mathbf{v}_m \end{pmatrix}$$

The m columns of V represent the velocities of a system of m particles.
In Einstein's special relativity theory, $m = 1$ (and $n = 3$).

An application of the Lorentz bi-boost of signature $(m, n) = (2, 3)$ to a system of $m = 2$ particles yields entanglement.

$$\begin{pmatrix} T' \\ X' \end{pmatrix} = \begin{pmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \\ \mathbf{x}'_1 & \mathbf{x}'_2 \end{pmatrix} := B_c(V) \begin{pmatrix} T \\ X \end{pmatrix} = B_c(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}$$

$$= \left\{ B_\infty(V) + \frac{1}{c^2} E(V) \right\} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix} + \frac{1}{c^2} E(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}, \tag{34}$$

The relativistic squared bi-norm of each subparticle of the $(2,3)$-particle (T, X) remains invariant under the application in (34) of the bi-boost $B_c(V) \in \mathrm{SO_c}(2,3)$, that is,

$$\begin{aligned}(t'_{11})^2 + (t'_{21})^2 - c^{-2}(\mathbf{x}'_1)^2 &= t_1^2 - c^{-2}\mathbf{x}_1^2 \\ (t'_{12})^2 + (t'_{22})^2 - c^{-2}(\mathbf{x}'_2)^2 &= t_2^2 - c^{-2}\mathbf{x}_2^2\,,\end{aligned} \tag{35}$$

where we use the notation $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x}$ for vectors $\mathbf{x} \in \mathbb{R}^n$.
Owing to the presence of emtanglement of times and spaces, the invariance in (35) exhibits a Lorentz symmetry violation.
Indeed, here the symmetry group is $\mathrm{SO_c}(2,3)$ rather than the symmetry group $\mathrm{SO_c}(1,3)$ of special relativity theory.

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Moreover, the relativistic bi-inner product of the two subparticles $(\mathbf{t}_k, \mathbf{x}_k)$, $k = 1, 2$, of the $(2, 3)$-particle (T, X) remains invariant under a bi-boost application, as well,

$$\begin{aligned} t'_{11}t'_{12} + t'_{21}t'_{22} - c^{-2}\mathbf{x}'_1 \cdot \mathbf{x}'_2 &= t_1 0 + 0 t_2 - c^{-2}\mathbf{x}_1 \cdot \mathbf{x}_2 \\ &= -c^{-2}\mathbf{x}_1 \cdot \mathbf{x}_2 \end{aligned} \qquad (36)$$

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The talk is based on topics from my recently published book:

"Beyond Pseudo-Rotations in Pseudo-Euclidean Spaces:
An introduction to the theory of
bi-gyrogroups and bi-gyrovector spaces"

Elsevier/Academic Press, 2018.

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Thank You