Geometry, Integrability and Quantization

Upper Bounds of Some Special Zeros of Functions in the Selberg Class

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- analytic properties of zeta and L-functions and
- geometric, algebraic, arithmetic,... properties of the object to which the function is associated.

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- The sum on the left is taken over all prime powers, and the sum on the right is taken over the non-trivial zeros of Riemann zeta function.

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- In 2010, Smajlović [6] and in 2011, Odžak and Smajlović [2] prove the explicit formula for functions in the Selberg class and its generalizations.
- The explicit formula motivates the study of properties of certain special zeros of functions in the Selberg class.

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- Special values, especially the value at the central point $s = 1/2$ is an important property and subject of intensive study.
- It arose in connection with the Birch and Swinnerton-Dyer conjecture.

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- an Euler product,
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which satisfy the following axioms:

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(1) (*Dirichlet series*) The Dirichlet series converges absolutely for $\Re(s) > 1$.

(2) (*Analytic continuation*) There exists an integer $m \geq 0$ such that the function $(s-1)^m F(s)$ is entire function of finite order. The smallest such number is denoted by m_F and called the *polar order* of F.

(3) (*Functional equation*) The function F satisfies the functional equation $\Phi_F(s) = w\Phi_F(1-\bar{s})$, where

$$\Phi_F(s) = F(s) Q_F^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j),$$

with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\Re(\mu_j) \geq 0, j = 1, \ldots, r$.

(4) (*Ramanujan hypothesis*) For every $\epsilon > 0$ we have $a_F(n) \ll n^\epsilon$.

(5) (*Euler product*) $\log F(s) = \sum_{n=1}^\infty \frac{b_F(n)}{n^s}$, where $b_F(n) = 0$ for all $n \neq p^m$ with $m \geq 1$ and p prime, and $b_F(n) \ll n^\theta$ for some $\theta < 1/2$.

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- $\zeta_K(s)$, the Dedekind zeta function of an algebraic number field K.
- $L_K(s, \chi)$, the Hecke L-function to a primitive Hecke character χ mod $\mathfrak{f}$ where $\mathfrak{f}$ is an ideal of the ring of integers of K .

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- Clearly, $\mathcal{S}^\sharp \supset \mathcal{S}$.

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  with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\Re(\mu_j) > -\frac{1}{4}, \Re(\lambda_j + 2\mu_j) > 0, j = 1, \ldots, r$ and
  $$\Phi_F^c(s) = (s-1)^{m_F} s^{m_F} \Phi_F(s).$$
  - (5') (*Euler sum*) The logarithmic derivative of the function F possesses a Dirichlet series representation $\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \frac{c_F(n)}{n^s}$, converging absolutely for $\Re s > 1$.

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- [2][Proposition 2.1] The class $\mathcal{S}$ is a subclass of $\mathcal{S}^{\sharp\flat}$.

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  - Degree of F (n=0): $H_F(0) = 2\sum_{j=1}^{r} \lambda_j = d_F$
  - Conductor:

$$q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}. \tag{1}$$

Explicit formula for functions in $\mathcal{S}^{\sharp\flat}$

The crucial tool for deriving main results is the explicit formula for functions in the Selberg class and its generalizations, applied to suitably constructed test functions.

Explicit formula for functions in $\mathcal{S}^{\sharp\flat}$

Theorem 1.

[6, Theorem 3.1], [2, Proposition 2.2] Let a regularized function G satisfy the following conditions:

1. $G \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$.
2. $G(x)e^{(1/2+\epsilon)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$, for some $\epsilon > 0$.
3. $G(x) + G(-x) - 2G(0) = O(|\log|x||^{-\alpha})$, as $x \to 0$, for some $\alpha > 2$.

Let $g(x) = G(-\log x)$, for $x > 0$, $G_j(x) = G(x)\exp\left(\frac{ix\Im\mu_j}{\lambda_j}\right)$ and $Z(F)$ the set of all non-trivial zeros of $F \in \mathcal{S}^{\sharp\flat}$.

Explicit formula for functions in $\mathcal{S}^{\sharp\flat}$

Theorem 1. continued

Then, the formula

$$\lim_{T\to\infty} \sum_{\substack{\rho\in Z(F)\\ |\Im\rho|\leq T}} ord(\rho) M_{\frac{1}{2}} g(\rho)$$
$$= m_F M_{\frac{1}{2}} g(0) + m_F M_{\frac{1}{2}} g(1)$$
$$- \sum_n \frac{c_F(n)}{n^{\frac{1}{2}}} g(n) - \sum_n \frac{\overline{c_F}(n)}{n^{\frac{1}{2}}} g(1/n) + 2G(0)\log Q_F$$
$$+ \sum_{j=1}^{r} \int_0^\infty \left[\frac{2\lambda_j G_j(0)}{x} - \frac{\exp((1-\frac{\lambda_j}{2} - \Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}} (G_j(x) + G_j(-x)) \right] e^{\frac{-x}{\lambda_j}}\, dx \tag{2}$$

holds true for an arbitrary function $F \in \mathcal{S}^{\sharp\flat}$, where $M_{\frac{1}{2}} g$ denotes the translate by $1/2$ of the Mellin transform of the function g.

Explicit formula for functions in $\mathcal{S}^{\sharp\flat}$

Corollary 1

Let G be an even regularized function satisfying conditions of Theorem 1. then, the formula

$$\lim_{T\to\infty} \sum_{\substack{\rho\in Z(F)\\ |\Im\rho|\leq T}} ord(\rho) M_{\frac{1}{2}} g(\rho)$$

$$= m_F M_{\frac{1}{2}} g(0) + m_F M_{\frac{1}{2}} g(1) - 2\sum_{n} \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g(1/n) + 2G(0)\log Q_F$$

$$+ 2\sum_{j=1}^{r} \int_0^\infty \left[\frac{2\lambda_j G(0)}{x} - \frac{\exp((1-\frac{\lambda_j}{2} - \Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}} G(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right)\right] e^{\frac{-x}{\lambda_j}} dx \tag{3}$$

holds true for an arbitrary function $F \in \mathcal{S}^{\sharp\flat}$.

Multiplicity of the zero at central point

Assuming generalised Riemann hypothesis (GRH) we prove the following

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Theorem 2.

Let R be the multiplicity of the eventual zero at the central point $1/2$ of function $F \in \mathcal{S}^{\sharp\flat}$ such that $\Re(c_F(n)) \geq 0$ and let

$$B(F) = 2\sum_{j=1}^{r} \lambda_j \left(\Re\left(\Psi\left(\frac{\lambda_j}{2} + \mu_j \right)\right) - \log(2\pi\lambda_j)\right).$$

a) If $q_F > e$, then

$$R \leq \frac{(4m_F + 1)\log q_F + B(F)}{2\log\log q_F}.$$

Multiplicity of the zero at central point

Theorem 2. continued

b) If $0 < q_F \leq e$, then

i) $R = 0$, for $m_F = 0$,

ii)

$$R \leq \frac{4m_F e^{W\left(\frac{B(F)+1}{4em_F}\right)+1} + B(F) + 1}{2\left(W\left(\frac{B(F)+1}{4em_F}\right) + 1\right)},$$

for $4m_F + B(F) + 1 > 0$,

where m_F is the polar order of F, q_F is the conductor of F, λ_j, μ_j are given as in axiom *(3')* and W denotes the Lambert function.

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As an immediate consequence of the above theorem, in the case when the conductor of function F is small, we get the following

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Corollary 2

Let $F \in \mathcal{S}^{\sharp\flat}$ be such that $\Re(c_F(n)) \geq 0$. Assume also that the conductor, q_F of F is less then or equal to e and that F is holomorphic. Then, $F(1/2) \neq 0$, i.e. F is non-vanishing at the central point.

Multiplicity of the zero at central point

Remark

From the proof of the Theorem 2. it is easy to see that the statement of theorem holds true under slightly less restrictive assumptions on $\Re(c_F(n))$. Namely, it is sufficient to assume that

$$\sum_n \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g_T(1/n) \geq 0.$$

Multiplicity of the zero at central point

Since automorphic L-functions $L(s,\pi)$ attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$, belongs to the class $\mathcal{S}^{\sharp\flat}$,

Multiplicity of the zero at central point

Since automorphic L-functions $L(s, \pi)$ attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$, belongs to the class $\mathcal{S}^{\sharp\flat}$, we can apply result of Theorem 2. to get the following

Multiplicity of the zero at central point

Corollary 3

Let R be the multiplicity of the eventual zero at the central point $1/2$ of $L(s,\pi)$ such that $\Re(c_n(\pi)) \geq 0$ and let

$$B(L) = \sum_{j=1}^{N} \Re\left(\Psi\left(\frac{1}{4} + \frac{1}{2}\kappa_j(\pi)\right)\right) - N \log \pi.$$

a) If $Q(\pi) > e$ then

$$R \leq \frac{(4m_L + 1)\log Q(\pi) + B(L)}{2\log\log Q(\pi)}.$$

where W denotes the Lambert function.

Multiplicity of the zero at central point

Corollary 3- continued

b) If $0 < Q(\pi) \leq e$ then

- i) $R = 0$, when $N > 1$ or $N = 1$ and $\pi \neq Id$.
- ii)

$$R \leq \frac{4m_L e^{W\left(\frac{1-\gamma-\pi/2-\log 8\pi}{4e}\right)+1} + 1 - \gamma - \pi/2 - \log 8\pi}{2\left(W\left(\frac{1-\gamma-\pi/2-\log 8\pi}{4e}\right)+1\right)}.$$

where W denotes the Lambert function.

Multiplicity of the zero at central point

Specially, if $L(s,\pi) \neq \zeta(s)$ is automorphic L-function with analytic conductor $Q(\pi)$ less than or equal to e, then $L(s,\pi)$ is non-vanishing at central point $s = 1/2$.

Location of the first zero with positive imaginary part

In this section we provide an upper bound for the height of the first zero with positive imaginary part of the function F in $\mathcal{S}^{\sharp\flat}$ such that $\Re(c_F(n)) \geq 0$ for all $n \in \mathbb{N}$.

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Theorem 3.

Let h be the height of the first zero with imaginary part different from zero of the function $F \in \mathcal{S}^{\sharp\flat}$. Assume that F satisfies axiom *(5)* of the Selberg class and $\Re(c_F(n)) \geq 0$. Then, for $q_F > e$ we have the bound

$$h \leq \max\left\{ \frac{16\sqrt{2}\left[(4m_F+1)\log q_F + B(F)\right]}{\pi \log q_F \log\log q_F}, \frac{(2\theta+1)\pi}{\sqrt{2}\log[\log q_F/16(K_F+\delta)]} \right\}.$$

Here q_F is the conductor of F, m_F is the polar order of F, $B(F)$ is given in Theorem 2, $K_F = \frac{C_F}{2\theta+1}$, $\theta < 1/2$ stemmed from axiom *(5)* of the Selberg class and $\delta > 0$.

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In the case when $F \in \mathcal{S}$ with non-negative coefficients, we can get sharper upper bound for the height of the first zero of F with positive imaginary part, as stated in the following

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Theorem 4.

Let h be the height of the first zero with imaginary part different from zero of the function $F \in \mathcal{S}$ and $F(1+it) \neq 0$ for all $t \in \mathbb{R}$ such that $a_F(n) \geq 0$ for all $n \in \mathbb{N}$. Then, for $q_F > e$ we have the bound $h \leq \max\left\{ \frac{16\sqrt{2}\left[(4m_F+1)\log q_F + B(F)\right]}{\pi \log q_F \log\log q_F}, \frac{\pi}{\sqrt{2}\log[\log q_F/16(m_F+\tau)]} \right\}$, where q_F is as in *(1)*, m_F is defined in axiom *(2)* of the Selberg class, $B(F)$ is given in Theorem 2 and $\tau > 0$.

Location of the first zero with positive imaginary part

Assuming *GRH* for automorphic L-functions and applying these results for $L(s, \pi) \in \mathcal{S}$ we prove

Location of the first zero with positive imaginary part

Corollary 4

Let h be the height of the first zero with imaginary part different from zero of the function $L(s,\pi)$. Assume that $L(s,\pi)$ satisfies axiom *(5)* of the Selberg class and $\Re(c_n(\pi)) \geq 0$, where $c_n(\pi) = b_n(\pi)\log n$. Then, for $Q(\pi) > e$ we have the bound

$$h \leq \max\left\{ \frac{16\sqrt{2}\left[\log Q(\pi) + B(L)\right]}{\pi \log Q(\pi) \log\log Q(\pi)}, \frac{(2\theta+1)\pi}{\sqrt{2}\log[\log Q(\pi)/16(K_L+\delta)]} \right\}.$$

Here m_L is defined in axiom *(2)* of the Selberg class, $B(L)$ is given in *Corollary 3*, $K_L = \frac{C_L}{2\theta+1}$, $\theta < 1/2$ and $\delta > 0$.

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Thank you!