Recent results in the Hamiltonian reduction approach to integrable many-body systems

Poisson–Lie analogues of spin Sutherland models

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Kazhdan, Kostant and Sternberg (1978): Derived the trigonometric Sutherland model by Hamiltonian reduction of free motion on $T^*\mathrm{U}(n)$.

Analogous reduction of cotangent bundle of any compact simple Lie group, at arbitrary moment map value, leads to spin Sutherland model.

LF and Klimčík (2009): Poisson-Lie analogue of the KKS reduction of $T^*\mathrm{U}(n)$ gives the real, trigonometric Ruijsenaars–Schneider model.

In this talk, based on arXiv:1809.01529, I present generalization of spin Sutherland models that descend from Poisson–Lie analogue of T^*G for any compact simple Lie group G.

Plan: I start with a recall of the reduction of T^*G, then present its Poisson–Lie analogue. I shall finish with comments on related results, consequences, generalizations and open problems.

Consider realification of complex simple Lie algebra: $\mathcal{G}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}$.

Compact: $\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{(E_\alpha - E_{-\alpha}), \mathrm{i}(E_\alpha + E_{-\alpha}), \mathrm{i}T_{\alpha_k} \mid \alpha \in \Phi^+, \alpha_k \in \Delta\}$

'Borel': $\mathcal{B} = \mathrm{span}_{\mathbb{R}}\{E_\alpha, \mathrm{i}E_\alpha, T_{\alpha_k} \mid \alpha \in \Phi^+, \alpha_k \in \Delta\}$

Isotropic subalgebras w.r.t. bilinear form

$$\langle X, Y\rangle := \mathrm{Im}(X, Y), \ \forall X, Y \in \mathcal{G}^{\mathbb{C}}, \text{ with Killing form } (\ ,\ ) \text{ of } \mathcal{G}^{\mathbb{C}}.$$

Starting phase space: $M := T^*G \times \mathcal{O}$ with coadjoint orbit $\mathcal{O}$ of compact Lie group G. Natural Poisson maps

$$J_L : M \to \mathcal{G}^*, \quad J_R : M \to \mathcal{G}^*, \quad J_{\mathcal{O}} : M \to \mathcal{G}^*.$$

Reduced phase space: $M_{\mathrm{red}} := \mu^{-1}(0)/G$ with $\mu := J_L + J_R + J_{\mathcal{O}}$.

M_{red} contains dense open subset $M_{\mathrm{red}}^{\mathrm{reg}} = T^*\mathbb{T}^o \times \mathcal{O}_0/\mathbb{T}$,

where $\mathbb{T}^o$ is interior of a Weyl alcove in the maximal torus $\mathbb{T} < G$.

Using $\mathcal{G}^* \simeq \mathcal{G}$ and product map $\pi_G \times J_R \times J_{\mathcal{O}}$ identify

$$M \equiv G \times \mathcal{G} \times \mathcal{O} = \{(g, J, \xi)\}, \text{ symplectic form: } \omega = -d(J, g^{-1}dg) + \omega_{\mathcal{O}}.$$

Moment map μ generates 'conjugation action' of G:

$$A_\eta(g, J, \xi) = (\eta g \eta^{-1}, \eta J \eta^{-1}, \eta \xi \eta^{-1}), \quad \forall \eta \in G.$$

Every element of $\mu^{-1}(0)$ is G-equivalent to a triple (Q^{-1}, J, ξ) with Q from closure of $\mathbb{T}^o \subset \mathbb{T}$. Assuming that $Q = e^{\mathrm{i}q}$ is regular, one can solve the constraint, $e^{-\mathrm{i}q} J e^{\mathrm{i}q} - J = \xi$, as follows:

$$\xi = \sum_{\alpha \in \Phi^+} (\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha}), \quad J = -\mathrm{i}p + \sum_{\alpha \in \Phi^+} (J_\alpha E_\alpha - J_\alpha^* E_{-\alpha}),$$

where $\mathrm{i}p \in \mathcal{T}$ is arbitrary and $J_\alpha = \frac{\xi_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1}$. This gives the model

$$M_{\mathrm{red}}^{\mathrm{reg}} = \mathbb{T}^o \times \mathcal{T} \times (\mathcal{O}_0/\mathbb{T}) = \{(e^{\mathrm{i}q}, \mathrm{i}p, [\xi])\}, \quad \omega_{\mathrm{red}} = (dp \overset{\wedge}{,} dq) + \omega_{\mathcal{O}}^{\mathrm{red}}.$$

Free Hamiltonian $\mathcal{H} := -\frac{1}{2}(J, J)$ reduces to

$$\mathcal{H}_{\mathrm{Suth}}(e^{\mathrm{i}q}, p, [\xi]) = \frac{1}{2}(p, p) + \frac{1}{2} \sum_{\alpha > 0} \frac{1}{|\alpha|^2} \frac{|\xi_\alpha|^2}{\sin^2 \frac{\alpha(q)}{2}}.$$

In general, this represents a spin Sutherland model.

Sutherland dynamics is projection of 'free motion':

$$g(t) = g(0)\,\exp(tJ(0)), \quad J(t) = J(0), \quad \xi(t) = \xi(0).$$

The 'kinetic energy' $\mathcal{H} = -\frac{1}{2}(J,J)$ belongs to Abelian Poisson algebra $C_I(M) := J_R^*(C^\infty(\mathcal{G}^*)^G)$. The free motion is degenerately integrable, because $C_I(M)$ Poisson commutes with each element of the Poisson algebra $C_J(M)$ generated by the components of J_L, J_R and $J_{\mathcal{O}}$.

Generically, integrability is inherited under Hamiltonian reduction.

$\Big($ $\mathcal{G}$ and $\mathcal{B}$ yield two models of $\mathcal{G}^*$; $\mathcal{G} \ni \xi \Longleftrightarrow \tilde{\xi} \in \mathcal{B}$ via $(\xi, X) = \langle \tilde{\xi}, X\rangle$, $\forall X \in \mathcal{G}$. In terms of constrained spin variable $\tilde{\xi} = \sum_{\alpha\in\Phi^+} \tilde{\xi}_\alpha E_\alpha$

$$\mathcal{H}_{\mathrm{Suth}}(e^{\mathrm{i}q}, p, [\tilde{\xi}]) = \frac{1}{2}(p,p) + \frac{1}{8}\sum_{\alpha\in\Phi^+} \frac{1}{|\alpha|^2}\frac{|\tilde{\xi}_\alpha|^2}{\sin^2\frac{\alpha(q)}{2}}.$$

This will be convenient for comparison with the spin RS models. $\Big)$

Heisenberg double [Semenov-Tian-Shansky, Alekseev–Malkin]. Consider *real* Lie group $G^{\mathbb{C}}$ and its subgroups G and B, corresponding to $\mathcal{G}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}$. Every element $K \in G^{\mathbb{C}}$ admits Iwasawa decompositions

$$K = b_L g_R^{-1} = g_L b_R^{-1}, \quad b_L, b_R \in B,\ g_L, g_R \in G.$$

$G^{\mathbb{C}}$ is equipped with symplectic form

$$\Omega_+ = \frac{1}{2}\left\langle db_L b_L^{-1} \overset{\wedge}{,} dg_L g_L^{-1}\right\rangle + \frac{1}{2}\left\langle db_R b_R^{-1} \overset{\wedge}{,} dg_R g_R^{-1}\right\rangle .$$

Define maps Λ_L, Λ_R from $G^{\mathbb{C}}$ to B and maps Ξ_L, Ξ_R from $G^{\mathbb{C}}$ to G by

$$\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R.$$

These are Poisson maps w.r.t. Poisson structure associated with Ω_+ and multiplicative Poisson structures on B and on G.

G acts on B by dressing action, $\mathrm{Dress}_\eta(b) := \Lambda_L(\eta b)$, and dressing orbits $(\mathcal{O}_B, \Omega_{\mathcal{O}_B})$ are symplectic leaves in B.

Reduction of free system on phase space $(\mathcal{M}, \Omega)$:

$$\mathcal{M} := G^{\mathbb{C}} \times \mathcal{O}_B = \{(K,S) \mid K \in G^{\mathbb{C}},\ S \in \mathcal{O}_B\}, \quad \Omega = \Omega_+ + \Omega_{\mathcal{O}_B}.$$

$C_I(\mathcal{M}) := \Lambda_R^*(C^\infty(B)^G)$ gives an Abelian Poisson algebra. Hamiltonian $\Lambda_R^*(h) \in C_I(\mathcal{M})$ generates 'free' flow

$$g_R(t) = \exp\left[t d^L h(b_R(0))\right] g_R(0),\ b_L(t) = b_L(0),\ b_R(t) = b_R(0),\ S(t) = S(0).$$

This is a degenerately integrable system, since all functions of b_L, b_R and S are conserved ($K = b_L g_R^{-1} = g_L b_R^{-1}$). They form the ring $C_J(\mathcal{M})$.

Here, derivative $d^L h(b) \in \mathcal{G}$ of any $h \in C^\infty(B)$ is defined by relation $\left\langle d^L h(b), X \right\rangle := \frac{d}{ds}\Big|_{s=0} h(\exp(sX)b)$ for all $X \in \mathcal{B}$ and $b \in B$.

A Poisson action of G on $\mathcal{M}$ is generated by non-Abelian moment map

$$\Lambda := \Lambda_L \Lambda_R \Lambda_{\mathcal{O}_B} : \mathcal{M} \to B \equiv G^*, \quad \text{for which} \quad \Lambda(K,S) = b_L b_R S.$$

$$\eta \in G \text{ acts by } A_\eta(K,S) = (\eta K \Xi_R(\eta b_L), \mathrm{Dress}_{\Xi_R(\eta b_L b_R)^{-1}}(S)).$$

$$C_I(\mathcal{M}) \text{ and } C_J(\mathcal{M})^G \text{ descend to } \mathcal{M}_{\mathrm{red}} := \Lambda^{-1}(e)/G.$$

Maximal torus $\mathbb{T} < G$ acts on $\mathcal{O}_B$ by conjugations. Writing $S \in \mathcal{O}_B$ as $S = S_0 S_+$ with $S_0 \in B_0$, $S_+ \in B_+$, this action has moment map $S \mapsto \log(S_0) \in \mathcal{B}_0$. Imposing $S_0 = e$, we obtain reduced dressing orbit

$$\mathcal{O}_B^{\text{red}} = (\mathcal{O}_B \cap B_+)/\mathbb{T}.$$

We focus on dense open submanifold $\mathcal{M}^{\text{reg}} := \Xi_R^{-1}(G^{\text{reg}}) \subset \mathcal{M}$, i.e., we assume that in $K = b_L g_R^{-1}$ we have $g_R \in G^{\text{reg}}$.

Main Theorem. *The open dense subset $\mathcal{M}^{\text{reg}}_{\text{red}} = (\Lambda^{-1}(e) \cap \mathcal{M}^{\text{reg}})/G$ of $\mathcal{M}^{\text{red}}$ can be identified with*

$$T^*\mathbb{T}^o \times \mathcal{O}_B^{\text{red}},$$

where $\mathbb{T}^o \subset \mathbb{T}$ is open Weyl alcove and $\mathcal{O}_B^{\text{red}}$ is reduced dressing orbit. The reduced symplectic structure reads $\Omega_{\text{red}} = \Omega_{T^*\mathbb{T}^o} + \Omega^{\text{red}}_{\mathcal{O}_B}$.

Crux of proof: $\mathcal{Z} := \{(K, S) \mid \Lambda(K, S) = e,\ \Xi_R(K) \in \mathbb{T}^o\}$ meets every G-orbit, and $\mathcal{M}^{\text{reg}}_{\text{red}} = \mathcal{Z}/\mathbb{T}$. With $b_R = b_0 b_+ = e^p b_+$ and $g_R = Q$, the constraint becomes

$$Q^{-1} b_+^{-1} Q b_+ S = e.$$

$b_0 = e^p \in B_0$, $Q \in \mathbb{T}^o$ and $S = S_+ \in \mathcal{O}_B \cap B_+$ are arbitrary, and b_+ is determined by Q and S_+.

Some notations: Let θ denote the Cartan involution of $\mathcal{G}^{\mathbb{C}} = \mathcal{G} + \mathrm{i}\mathcal{G}$, and Θ the Cartan involution of $G^{\mathbb{C}}$. We write

$$X^\dagger := -\theta(X), \quad K^\dagger := \Theta(K^{-1}) \quad \text{for} \quad X \in \mathcal{G}^{\mathbb{C}},\ K \in G^{\mathbb{C}}.$$

Defining $\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G^{\mathbb{C}}$, one has G-equivariant diffeomorphism

$$B \ni b \mapsto bb^\dagger \in \mathfrak{P}, \quad \text{with } G \text{ acting on } \mathfrak{P} \text{ by conjugations.}$$

In this way $C^\infty(B)^G$ is turned into $C^\infty(\mathfrak{P})^G$, which is generated by the restrictions of the characters χ_ρ of the fundamental irreps of $G^{\mathbb{C}}$.

The *'main reduced Hamiltonians'* descend from the characters. We define $H^\rho \in C^\infty(\mathcal{M})^G$ by

$$H^\rho(K, S) := \mathrm{tr}_\rho(b_R b_R^\dagger) := c_\rho \mathrm{tr}(\rho(b_R b_R^\dagger)) \quad \text{with} \quad K = g_L b_R^{-1}.$$

(The constant c_ρ is chosen so that $c_\rho \mathrm{tr}\left(\rho(E_\alpha)\rho(E_{-\alpha})\right) = 2/|\alpha|^2$, and we put $\mathrm{tr}_\rho(XYZ) := c_\rho \mathrm{tr}(\rho(X)\rho(Y)\rho(Z))$ etc.)

Interpretation as spin RS model: Constraint $Q^{-1}b_+^{-1}Qb_+ = S_+^{-1}$,

$$S_+ = e^{\sigma}, \quad b_+ = e^{\beta}, \quad \sigma = \sum_{\alpha>0} \sigma_\alpha E_\alpha, \quad \beta = \sum_{\alpha>0} \beta_\alpha E_\alpha, \quad Q = e^{\mathrm{i}q}.$$

Baker-Campbell-Hausdorff formula gives

$$\exp(\beta - Q^{-1}\beta Q - \frac{1}{2}[Q^{-1}\beta Q, \beta] + \cdots) = \exp(-\sigma).$$

β_α can be expressed in terms of σ and $e^{\mathrm{i}q}$:

$$\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} + \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q})\sigma_{\varphi_1}\dots\sigma_{\varphi_k},$$

where $\alpha = \varphi_1 + \cdots + \varphi_k$ and $f_{\varphi_1,\dots,\varphi_k}$ depends rationally on $e^{\mathrm{i}q}$.

Therefore $H^{\rho}_{\mathrm{red}} = \mathrm{tr}_\rho(e^p b_+ b_+^\dagger e^p)$ can be expanded as

$$H^{\rho}_{\mathrm{red}}(e^{\mathrm{i}q}, p, [\sigma]) = \mathrm{tr}_\rho\left(e^{2p}\left(\mathbf{1}_\rho + \frac{1}{4}\sum_{\alpha>0}\frac{|\sigma_\alpha|^2 E_\alpha E_{-\alpha}}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*)\right)\right).$$

This can be called a spin RS type Hamiltonian.

By expanding e^{2p},

$$H^{\rho}_{\mathrm{red}}(e^{\mathrm{i}q}, p, [\sigma]) = \mathrm{dim}_\rho + 2\mathrm{tr}_\rho(p^2) + \frac{1}{2}\sum_{\alpha>0} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*, p).$$

Leading term of $\frac{1}{4}(H^{\rho}_{\mathrm{red}} - \mathrm{dim}_\rho)$ matches spin Sutherland Hamiltonian $\mathcal{H}_{\mathrm{Suth}}(e^{\mathrm{i}q}, p, [\tilde{\xi}])$.

Poisson brackets of functions of spin variables follow from

$$\{\tilde{\xi}^i, \tilde{\xi}^j\}_{\mathcal{G}^*}(\tilde{\xi}) = \langle [Y^i, Y^j], \tilde{\xi}\rangle, \quad \{\sigma^i, \sigma^j\}_{\mathrm{B}}(e^\sigma) = \langle [Y^i, Y^j], \sigma\rangle + \mathrm{o}(\sigma),$$

where $\tilde{\xi}^i = \langle \tilde{\xi}, Y^i\rangle$ for a basis $\{Y^i\}$ of $\mathcal{T}^\perp \subset \mathcal{G}$ and similarly for σ.

Elements of $C_I(\mathcal{M}) = \Lambda^*_R(C^\infty(B)^G)$ descend to G-invariant functions of 'Lax matrix' $L(e^{\mathrm{i}q}, p, \sigma) := e^p b_+ b_+^\dagger e^p$. In any representation,

$$L(e^{\mathrm{i}q}, p, \sigma) = 1 + 2p + \sum_{\alpha>0} \left(\frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} E_\alpha + \frac{\sigma^*_\alpha}{e^{\mathrm{i}\alpha(q)} - 1} E_{-\alpha} \right) + \mathrm{o}(\sigma, \sigma^*, p).$$

This matches the Sutherland Lax matrix. In conclusion, our models are generalizations of the spin Sutherland models.

Explicit formulas for $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$: Now parametrize $b \in B$ by its matrix elements. With $b_R = e^p b$, we can solve the constraint

$$Q^{-1} b Q = bS,$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_n) \in \mathbb{T}^o$, $S \in B_+$ is the constrained 'spin' variable and b is an unknown upper triangular matrix with unit diagonal.

Using the notation $\mathcal{I}_{a,a+j} = \frac{1}{Q_{a+j}Q_a^{-1} - 1}$, we have $b_{a,a+1} = \mathcal{I}_{a,a+1} S_{a,a+1}$, and, for $k = 2, \ldots, n-a$, the matrix element $b_{a,a+k}$ equals

$$\mathcal{I}_{a,a+k} S_{a,a+k} + \sum_{\substack{m=2,\ldots,k \\ (i_1,\ldots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=k}} \prod_{\alpha=1}^{m} \mathcal{I}_{a,a+i_1+\cdots+i_\alpha} S_{a+i_1+\cdots+i_{\alpha-1},a+i_1+\cdots+i_\alpha}.$$

The reduction of $H = \mathrm{tr}(b_R b_R^\dagger)$ gives

$$H_{\mathrm{red}}(e^{\mathrm{i}q}, p, [S]) = \sum_{a=1}^{n} e^{2p_a} + \frac{1}{4}\sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|S_{a,a+k}|^2}{\sin^2((q_{a+k}-q_a)/2)} + \mathrm{o}_2(S, S^\dagger).$$

The minimal dressing orbit of $\mathrm{SU}(n)$ (and a canonical transformation) results in the standard (spinless) real, trigonometric RS model.

Reduced equations of motion and solutions: Define $H \in C_I(\mathcal{M})$ by $H(K,S) = h(b_R)$, and denote $(d^L h)(b_R) =: \mathcal{V}(L)$ with $L := b_R b_R^\dagger$. The Hamiltonian vector field of H on $\mathcal{M}$ gives

$$\dot{g}_R = \mathcal{V}(L) g_R, \quad \dot{b}_R = 0, \quad \dot{S} = 0 \qquad (K = b_L g_R^{-1} = g_L b_R^{-1}).$$

In the 'diagonal gauge' $\mathcal{Z}$, where $g_R = Q \in \mathbb{T}^o$, one recovers S from Q and $L = b_R b_R^\dagger$ via $S = b_R^{-1} Q^{-1} b_R Q$.

Decompose any $Y \in \mathcal{G}$ as $Y = Y_{\mathcal{T}} + Y_\perp$, using $\mathcal{G} = \mathcal{T} + \mathcal{T}^\perp$. Introduce the dynamical r-matrix $\mathcal{R}(Q)$ that acts as zero on the Cartan subalgebra $\mathcal{T}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ and acts on the span of the root vectors by

$$\mathcal{R}(Q) = \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id})(\mathrm{Ad}_Q - \mathrm{id})^{-1}.$$

Proposition. *The projection of the Hamiltonian vector field to the 'diagonal gauge' reads*

$$\dot{Q} = \mathcal{V}_{\mathcal{T}}(L) Q, \qquad \dot{L} = [Y_{\mathcal{T}} + (\mathcal{R}(Q) + 1/2)\mathcal{V}_\perp(L), L],$$

where $Y_{\mathcal{T}}$ is arbitrary. The solutions are obtained by diagonalization:

$$Q(t) = \eta(t) \exp(t\mathcal{V}(L(0))) Q(0) \eta(t)^{-1} \quad \text{with} \quad \eta(t) \in G,$$

and then $L(t) = \eta(t) L(0) \eta(t)^{-1} = n_+(t) e^{2p(t)} n_+(t)^\dagger$, with $n_+(t) \in B_+$.

Constants of motion and integrability

Poisson algebra of integrals of free motion, $C_J(\mathcal{M})$, consists of all functions of b_L, b_R and S, and $C_J(\mathcal{M})^G$ suffices for degenerate integrability of reduced system. Particular G-invariant constants of motion are

$$\mathcal{F}(K,S) = \mathrm{tr}_\rho\Big(\mathcal{P}(b_R b_R^\dagger, g_R^{-1} b_R b_R^\dagger g_R)\Big), \quad (g_R^{-1} b_R b_R^\dagger g_R = b_L^{-1}(b_L^{-1})^\dagger),$$

where $\mathcal{P}$ is any non-commutative polynomial. In the 'diagonal gauge', these give

$$\mathcal{F}_{\mathrm{red}}(Q,L) = \mathrm{tr}_\rho\left(\mathcal{P}(L, Q^{-1}LQ)\right).$$

Spectral parameter dependent Lax matrix generates special integrals

$$\mathcal{L}(\lambda) := L + \lambda Q^{-1} L Q.$$

Reduced Hamiltonian vector field of $H = \Lambda_R^*(h) \in C_I(\mathcal{M})$ implies

$$\dot{\mathcal{L}}(\lambda) = [Y_{\mathcal{T}} + (\mathcal{R}(Q) + 1/2)\mathcal{V}_\perp(L), \mathcal{L}(\lambda)].$$

The reduced system is 'obviously' integrable in every reasonable sense.

Alternative construction: Poisson reduction

Instead of symplectic reduction, one may simply take the quotient of the unreduced phase space by the G-action.

In the $G = \mathrm{U}(n)$ case, the functions on the quotient can be identified with $\mathbb{T}^n$-invariant (and Weyl-invariant) functions on the gauge slice

$$\{(Q, L) \mid Q \in \mathbb{T}^n_{\mathrm{reg}},\ L \in \mathrm{i}\mathfrak{u}(n)\}.$$

The respective quotients of $T^*\mathrm{U}(n)$ and the Heisenberg double $\mathrm{GL}(n, \mathbb{C})$ lead to the **compatible** Poisson brackets:

$$\{f, h\}_1^{\mathrm{red}}(Q, L) = \langle D_1 f, d_2 h\rangle - \langle D_1 h, d_2 f\rangle + \langle L, [d_2 f, d_2 h]_{\mathcal{R}(Q)}\rangle,$$

and

$$\{f, h\}_2^{\mathrm{red}}(Q, L) = \langle D_1 f, L d_2 h\rangle - \langle D_1 h, L d_2 f\rangle + 2\langle L d_2 f, \mathcal{R}(Q)(L d_2 h)\rangle.$$

The derivatives $D_1 f \in \mathfrak{b}(n)_0$ and $d_2 f \in \mathfrak{u}(n)$ are evaluated at (Q, L), and we use $[X, Y]_{\mathcal{R}(Q)} := [\mathcal{R}(Q)X, Y] + [X, \mathcal{R}(Q)Y]$.

This gives the bi-Hamiltonian ‘spin Ruijsenaars–Sutherland’ hierarchy:

$$\{f, h_k\}_2 = \{f, h_{k+1}\}_1 \quad \text{with} \quad h_k := \frac{1}{k}\mathrm{tr}(L^k),\ k \in \mathbb{N}.$$

Concluding remarks

1. Degenerate integrability can be proved (generically) relying on the G-equivariant map $\mathcal{J} := \Lambda_L \times \Lambda_L\Lambda_R \times \Lambda_L\Lambda_R\Lambda_{\mathcal{O}_B} : \mathcal{M} \to B \times B \times B$.

2. Our trigonometric spin RS systems are related by analytic continuation to hyperbolic spin RS systems derived by L.-C. Li [2006] based on dynamical Poisson groupoids [used only the variables (q, L)]. They can be viewed as real forms of holomorphic spin RS systems descending from the Heisenberg double of $G^{\mathbb{C}}$, studied by Reshetikhin [2016].

3. Our reduced Hamiltonian flows are automatically complete. This framework accommodates action-angle duals, too.

4. We have a generalization involving twisted conjugations of G.

5. Compactified trigonometric spin RS models should arise from reductions of quasi-Hamiltonian double $G \times G$.

6. Gibbons–Hermsen type spin RS models can be obtained reducing

$$\mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n \text{ with constraint } \Lambda_L\Lambda_R\Lambda_1^{\mathbb{C}^n}\Lambda_2^{\mathbb{C}^n}\cdots\Lambda_k^{\mathbb{C}^n} = e^{\gamma}\mathbf{1}_n.$$

Currently studied with I. Marshall; related work by Chalykh and Fairon.

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