On Variational-Like Inequalities and Global Minimization Problem

Vsevolod Ivanov Ivanov

Technical University of Varna

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Short history notes

The object of investigation of this work are Stampacchia and Minty variational-like inequalities. Stampacchia variational inequality of differentiable type was developed by Stampacchia in 1964 and some subsequent works. The necessary optimality condition characterizes the connection between optimization problems and Stampacchia variational inequality. The study of Minty variational inequality originated from Minty in 1967.

Invex functions were introduced by Hanson in 1981. A lot of papers appeared since then. The book [Mishra, Giorgi, 2008] is a comprehensive survey of their properties and applications in optimization, economics, and engineering. The notion of invariant pseudomonotonicity was introduced by Yang, Yang, Teo in 2003 .

An abstract

It was found by Ivanov in 2008 which are the largest classes of functions such that the solution sets of each pair of the following problems coincide: Stampacchia variational inequality, Minty variational inequality, and the global minimization problem. We extend the results from [Ivanov, 2008] to variational-like inequalities. We obtain necessary and sufficient conditions, which ensure that all pairs of the solution sets of Stampacchia variational-like inequality, Minty variational-like inequality, and global minimization problem coincide. Our results are applications of the properties that every pseudoinvex function is prequasiinvex, and pseudoinvexity of a function is equivalent to invariant pseudomonotonicity of the gradient map.

Some preliminary definitions

In the sequel, $\mathbf{E}$ denotes a real linear space and $X \subset \mathbf{E}$ is a given set. We consider a finite-valued real function f, defined on X. Here $\mathbf{R}$ is the set of the reals and

$$\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$$

is the extended real line. Let $\overline{f} : \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ be the extension of f such that $\overline{f}(x) = +\infty$ for $x \in \mathbf{E} \setminus X$. We suppose additionally that $\eta : X \times X \to \mathbf{E}$ is a given map, which is called a kernel.

Definition

Recall that a function $f : X \to \mathbf{R}$ is said to be radially lower semicontinuous (in short, radially lsc), iff for every $x \in \mathbf{E}$, $u \in \mathbf{E}$ the function of one variable φ defined for every $t \in \mathbf{R}$ such that $x + tu \in X$ by $\varphi(t) = f(x + tu)$ is lower semicontinuous (in short, lsc).

Definition

Recall that the lower Dini directional derivative $f'_-(x, u)$ of f at $x \in X$ in direction $u \in \mathbf{E}$ is defined as an element of $\overline{\mathbf{R}}$ by

$$f'_-(x, u) = \liminf_{t \to 0^+} t^{-1}(\overline{f}(x + tu) - f(x)).$$

Definition (Moham, Neogy, 1995)

A set X is called invex with respect to a given kernel $\eta : X \times X \to \mathbf{E}$ iff

$$x + t\eta(y,x) \in X \quad \text{for all} \quad x, y \in X \quad \text{and every} \quad t \in [0,1].$$

Definition (Generalization of pseudoinvexity in [Hanson, 1981])

Recall that a function $f : X \to \mathbf{R}$ is said to be pseudoinvex on the set X with respect to a map $\eta : X \times X \to \mathbf{E}$ in terms of the lower Dini directional derivative iff

$$x,\, y \in X,\;\; f(y) < f(x) \quad \text{imply} \quad f'_-(x, \eta(y,x)) < 0.$$

Definition (Pini, 1991)

Let X be an invex set with respect to the map $\eta : X \times X \to \mathbf{E}$. Then a function $f : X \to \mathbf{R}$ is said to be prequasiinvex on X with respect to η iff

$$f(x + t\eta(y,x)) \le \max\left(f(x),\, f(y)\right)$$

for all $x,\, y \in X$ and $t \in [0,\, 1]$.

Definition (Moham, Neogy, 1995)

Let the set X be invex with respect to the kernel $\eta : X \times X \to \mathbf{E}$ and $f : X \to \mathbf{R}$. Then, it is said that $\eta : X \times X \to \mathbf{E}$ satisfies Condition C iff

$$\eta(x, x + t\eta(y,x)) = -t\eta(y,x),$$

$$\eta(y, x + t\eta(y,x)) = (1-t)\eta(y,x)$$

for all $x, y \in X$ and $t \in [0,1]$.

Definition (Yang, Yang, Teo, 2003)

Let the set X be invex with respect to the kernel $\eta : X \times X \to \mathbf{E}$ and $f : X \to \mathbf{R}$. Then, it is said that $\eta : X \times X \to \mathbf{E}$ satisfies Condition A iff

$$f(x + \eta(y,x)) \le f(y) \quad \text{for all} \quad x, y \in X.$$

Some properties of pseudoinvex functions under weaker hypotheses

First, we derive some connections between pseudoinvex functions and invariant pseudomonotone lower Dini derivatives.

Theorem

Let the set $X \subseteq \mathbf{E}$ be invex with respect to the kernel $\eta : X \times X \to \mathbf{E}$. Suppose that $f : X \to \mathbf{R}$ is a radially lsc function, which is pseudoinvex with respect to η and it satisfies Condition C. Then f is prequasiinvex with respect to η.

The claim that every pseudoinvex function is prequasiinvex appeared in the differentiable case in [Yang, Yang, Teo, 2003]. We prove it without the assumption that Condition B, introduced by these authors, is satisfied.

Definition

The lower Dini derivative f'_- of a function $f : X \to \mathbf{R}$ is said to be invariant pseudomonotone on X with respect to a map $\eta : X \times X \to \mathbf{E}$ iff for all $x, y \in X$ the following implication holds:

$$f'_-(x, \eta(y, x)) \geq 0 \quad \text{implies} \quad f'_-(y, \eta(x, y)) \leq 0.$$

Theorem

Let the set X be invex with respect to the given map $\eta : X \times X \to \mathbf{E}$, and $f : X \to \mathbf{R}$ be a radially lsc function. Suppose that f and η satisfy Conditions A and C. Then f is pseudoinvex with respect to η if and only if the lower Dini derivative of f is invariant pseudomonotone.

Nonlinear programming problem and Minty variational-like inequality

We consider the following variational-like inequality problem of Minty type:

$$\text{Find} \quad \bar{x} \in X \quad \text{such that} \quad f'_{-}(x, \eta(\bar{x}, x)) \leq 0, \quad \forall x \in X.$$

We denote its solution set by $M(f, X)$. Our aim is to obtain a relation between Minty variational-like inequality and nonlinear programming problem. We denote the set of global minimizers of f over X by $GM(f, X)$ and the following set by $A(x, y)$ for arbitrary $x \in X$, $y \in X$:

$$A(x, y) = \{z \in \mathbf{E} \mid z = x + t\eta(y, x), \quad t \in [0, 1]\}$$

Lemma

Let the set $X \subseteq \mathbf{E}$ *be invex with respect to the map* $\eta : X \times X \to \mathbf{E}$. *Suppose that* η *satisfies Condition C. Then the set* $A(x, y)$ *is invex for all* $x, y \in X$ *with respect to the same kernel* η.

Theorem

Let the set $X \subseteq \mathbf{E}$ *be invex with respect to the map* $\eta : X \times X \to \mathbf{E}$. *Suppose that* $f : X \to \mathbf{R}$ *is a radially lsc function. Suppose further that* f *and* η *satisfy Conditions A and C. Then the following statements are equivalent:*

(i) f *is prequasiinvex with respect to* η *on* X;

(ii) $GM(f, Y) \equiv M(f, Y)$ *for all invex subsets* $Y \subseteq X$;

(iii) $GM(f, A(x, y)) \equiv M(f, A(x, y))$ *for all* x, $y \in X$.

Stampacchia variational-like inequality and nonlinear programming problem

Another useful variational-like inequality is the variational-like inequality problem of Stampacchia type:

$$\text{Find} \quad \bar{x} \in X \quad \text{such that} \quad f'_-(\bar{x}, \eta(x, \bar{x})) \geq 0, \quad \forall x \in X.$$

We denote its solution set by $S(f, X)$.

Theorem

Let the set $X \subset \mathbf{E}$ *be invex with respect to the map* $\eta : X \times X \to \mathbf{E}$. *Suppose that* $f : X \to \mathbf{R}$ *is a function,* f *and* η *satisfy Conditions A and C. Then the following statements are equivalent:*

(i) f *is pseudoinvex on* X *with respect to* η *;*

(ii) $S(f, Y) \equiv GM(f, Y)$ *for all invex subsets* $Y \subseteq X$ *with respect to* η*;*

(iii) $S(f, A(x, y)) \equiv GM(f, A(x, y))$ *for all* $x, y \in X$.

Stampacchia and Minty variational-like inequalities

Theorem

Let the set X be invex with respect to the map $\eta : X \times X \to \mathbf{E}$. Suppose that $f : X \to \mathbf{R}$ is a radially lsc function, f and η satisfy Conditions A and C. Then the following statements are equivalent:

(i) *f is pseudoinvex on X with respect to η;*

(ii) *$S(f, Y) \equiv M(f, Y)$ for all invex subsets $Y \subseteq X$ with respect to η;*

(iii) *$S(f, A(x, y)) \equiv M(f, A(x, y))$ for all x, $y \in X$.*

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