FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

Maria Clara Nucci

University of Perugia & INFN-Perugia, Italy

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Lecture 2: Jacobi last multiplier and its properties

- The Jacobi last multiplier and its connection to first integrals and Lagrangians.
- The Jacobi last multiplier and its connection to Lie symmetries .
- Nonlocal symmetries as hidden symmetries: the role of Jacobi last multiplier.

Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788) Joseph-Louis Lagrange (1736-1813) wrote:

> The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field. (tr. by J.R. Maddox:)

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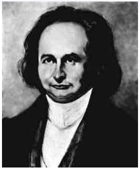
It is a joke, isn't it??!!

Jacobi last multiplier



(Jacobi, 1842-45)

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \qquad (\star)$$

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \qquad (\star\star)$$

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf$$

$\omega_i, (i = 1, \ldots, n-1)$ solutions of $(\star)$ ie first integrals of $(\star\star)$

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0 \Leftrightarrow \frac{\mathrm{d}\log(M)}{\mathrm{d}t} = -\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}$$

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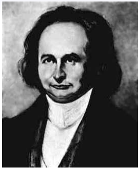

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IMPORTANT PROPERTY:

$$\frac{M_1}{M_2} = \text{First Integral}$$

Enter Lie

[Lie, 1874]

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \quad (\star\star)$$

If there exist $n-1$ symmetries of $(\star\star)$, say

$$\Gamma_i = \xi_{ij}\partial_{x_j}, \quad i = 1, n-1$$

then JLM is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}$$

Corollary: if $\exists\ M$=const, then Δ is a first integral.

How many Lagrangians does one know?

There is a link between a Jacobi Last Multiplier M and a Lagrangian L [Jacobi, 1842-45], [also in Whittaker, 1904].

Jacobi's Lectures on Dynamics (1884) are available in English: tr. by K. Balagangadharan, ed. by Biswarup Banerjee, Hindustan Book Agency (2009), available through AMS

For a second-order ODE the link is:

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Consequently a knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system.

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How many??

N.B.: For a single ODE of order $2n$ the link is $M^{1/n} = \dfrac{\partial^2 L}{\partial \left(q^{(n)}\right)^2}$

(Jacobi, *J. Reine Angew. Math. 29* (1845) p.364)

A very simple example

Let us consider the one-dimensional free particle $\ddot{x} = 0$, i.e.:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

Lie symmetry algebra $sl(3, \mathbb{R})$:

$$X_1 = xt\partial_t + x^2\partial_x, \quad X_2 = x\partial_t, \quad X_3 = t^2\partial_t + xt\partial_x, \quad X_4 = x\partial_x,$$
$$X_5 = t\partial_t, \quad X_6 = \partial_t, \quad X_7 = t\partial_x, \quad X_8 = \partial_x.$$

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For example $JLM_{48} = -1/\dot{x}$ by means of X_4 and X_8 such that:

$$\Delta_{48} = \det \begin{bmatrix} 1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 1 & 0 \end{bmatrix} = -x_2 \equiv -\dot{x}.$$

Ten Lagrangians

Ten different JLM and consequently as many Lagrangians:

$$M_{13} = -\frac{1}{(t\dot{x}-x)^3} \quad \Rightarrow \quad L_{1,3} = -\frac{1}{2t^2(t\dot{x}-x)} + \frac{\mathrm{d}g}{\mathrm{d}t}(t,x)$$

$$M_{15} = -\frac{1}{\dot{x}(t\dot{x}-x)^2} \quad \Rightarrow \quad L_{1,5} = \frac{\dot{x}}{x^2}\left(\log(t\dot{x}-x) - \log(\dot{x})\right)$$

$$M_{16} = \frac{1}{\dot{x}^2(t\dot{x}-x)} \quad \Rightarrow \quad L_{1,6} = \left(\frac{t\dot{x}}{x^2} - \frac{1}{x}\right)\left(\log(\dot{x}) - \log(t\dot{x}-x)\right)$$

$$M_{17} = -\frac{1}{(t\dot{x}-x)^2} \quad \Rightarrow \quad L_{1,7} = -\frac{1}{t^2}\log(t\dot{x}-x)$$

$$M_{18} = \frac{1}{\dot{x}(t\dot{x}-x)} \quad \Rightarrow \quad L_{1,8} = -\frac{\dot{x}}{x}\log(\dot{x}) - \left(\frac{1}{t} - \frac{\dot{x}}{x}\right)\log(t\dot{x}-x) + \frac{1}{t}(1+\log(x))$$

$$M_{62} = \frac{1}{\dot{x}^3} \quad \Rightarrow \quad L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \quad \Rightarrow \quad L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x} - x} \quad \Rightarrow \quad L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2}\right)(\log(t\dot{x} - x) - 1)$$

$$M_{48} = -\frac{1}{\dot{x}} \quad \Rightarrow \quad L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \quad \Rightarrow$$

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FINALLY, THE TRUE LAGRANGIAN

Lagrangians for biological models

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[Jacobi, J. Reine Angew. Math. 29 (1845)]

Lagrangians for biological systems with JLM

Given the following system:

$$\begin{aligned} \dot{u}_1 &= \phi_1(t, u_1, u_2) \\ \dot{u}_2 &= \phi_2(t, u_1, u_2) \end{aligned} \tag{2}$$

It was proven in [MCN & Tamizhmani, 2012] that if a Jacobi Last Multiplier M is determined for system (2) then its Lagrangian is:

$$L = \dot{u}_2 \int M \mathrm{d}u_1 - \dot{u}_1 \int M \mathrm{d}u_2 + V(t, u_1, u_2).$$

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If a Noether symmetry

$$\Gamma = \xi(t, u_1, u_2)\partial_t + \eta_1(t, u_1, u_2)\partial_{u_1} + \eta_2(t, u_1, u_2)\partial_{u_2} \tag{3}$$

exists for the Lagrangian L then a first integral of system (2) is

$$-\xi L - \frac{\partial L}{\partial \dot{u}_1}(\eta_1 - \xi \dot{u}_1) - \frac{\partial L}{\partial \dot{u}_2}(\eta_2 - \xi \dot{u}_2) + G(t, u_1, u_2). \tag{4}$$

Gompertz model

$$\begin{aligned} \dot{w}_1 &= w_1\left(A\log\left(\frac{w_1}{m_1}\right)+Bw_2\right) \\ \dot{w}_2 &= w_2\left(a\log\left(\frac{w_2}{m_2}\right)+bw_1\right). \end{aligned} \tag{5}$$

In order to simplify system (5) we introduce the change of variables

$$w_1 = m_1\exp(r_1), \qquad w_2 = m_2\exp(r_2) \tag{6}$$

and then system (5) becomes

$$\begin{aligned} \dot{r}_1 &= m_2 B\exp(r_2) + Ar_1 \\ \dot{r}_2 &= m_1 b\exp(r_1) + ar_2. \end{aligned} \tag{7}$$

It is easy to derive a Jacobi Last Multiplier for this system, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(M_{[r]}\right) = -(a+A) \Longrightarrow M_{[r]} = \exp[-(a+A)t] \tag{8}$$

We can transform system (7) into an equivalent second-order ODE by eliminating, say, r_2. In fact from the second equation in (7) one gets

$$r_2 = \log\left(\frac{\dot{r}_1 - Ar_1}{Bm_2}\right), \tag{9}$$

and the equivalent second-order equation in r_2 is
$\ddot{r}_1 = \left(bm_1 \exp(r_1) + a \log\left(\frac{\dot{r}_1 - Ar_1}{Bm_2}\right)\right)(\dot{r}_1 - Ar_1) + A\dot{r}_1.$
A Jacobi Last Multiplier for this equation can be obtained by calculating the Jacobian of the transformation between (r_1, r_2) and $(r_1, \dot{r}_1)$, i.e.

$$M_1 = M_{[r]} \frac{\partial(r_1, r_2)}{\partial(r_1, \dot{r}_1)} = \exp[-(a + A)t] \frac{1}{\dot{r}_1 - Ar_1}. \tag{10}$$

Then a Lagrangian can be easily obtained by a double integration, i.e.

$$L_1 = \exp[-(a + A)t]\Big((\dot{r}_1 - Ar_1)\log(\dot{r}_1 - Ar_1) + m_1 b \exp(r_1)$$
$$-ar_1 \log(Bm_2) - ar_1\Big) + \dot{F}(t, r_1).$$

Vito Volterra's last paper

Calculus of Variations and the Logistic Curve, Human Biology, 1939

Vito Volterra (1860-1940) wrote *"I have been able to show that the equations of the struggle for existence depend on a question of Calculus of Variations"*

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"In order to obtain this result, I have replaced the notion of population by that of quantity of life. In this manner I have also obtained some results by which dynamics is brought into relation to problems of the struggle for existence." The quantity of life X and the population N of a species are connected by the relation

$$N = \frac{\mathrm{d}X}{\mathrm{d}t}. \tag{11}$$

Thus Volterra takes a system of first-order equations and transform it into a system of second-order equations.

Volterra-Verhulst-Pearl equation

One of the equations Volterra considered is the Verhulst-Pearl equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N(\varepsilon - \lambda N) \tag{12}$$

that through (11) becomes

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = \frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda \frac{\mathrm{d}X}{\mathrm{d}t}\right). \tag{13}$$

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Equation (13) admits an eight-dimensional Lie symmetry algebra generated by the following operators:

$\Gamma_1 = \exp(\lambda X - \varepsilon t)\partial_t, \quad \Gamma_2 = \exp(\lambda X)\left(\partial_t + \frac{\varepsilon}{\lambda}\partial_X\right),$
$\Gamma_3 = \exp(-\lambda X + \varepsilon t)\partial_X, \quad \Gamma_4 = \exp(-\lambda X)\partial_X,$
$\Gamma_5 = \exp(\varepsilon t)\left(\frac{\lambda}{\varepsilon}\partial_t + \partial_X\right), \; \Gamma_6 = \partial_X, \; \Gamma_7 = \exp(-\varepsilon t)\partial_t, \; \Gamma_8 = \partial_t.$

Therefore the equation is linearizable

$$y = \exp(-\varepsilon t), \; u = \frac{1}{\lambda}\exp(\lambda X - \varepsilon t) \Rightarrow \frac{\mathrm{d}^2u}{\mathrm{d}y^2} = 0$$

Ten Lagrangians

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$$Lag_{14} = -\exp(\varepsilon t)\left(\frac{1}{\lambda}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + X\right),$$

$$Lag_{15} = \exp(-\lambda X)\left(\frac{1}{\varepsilon}\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \frac{1}{\varepsilon}\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{1}{\lambda}\right),$$

$$Lag_{17} = -\frac{1}{2\lambda\frac{\mathrm{d}X}{\mathrm{d}t}}\exp(2\varepsilon t - \lambda X),$$

$$Lag_{18} = \frac{1}{\varepsilon^2}\exp(\varepsilon t - \lambda X)\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\left(\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \varepsilon\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\right),$$

$$Lag_{23} = -\frac{1}{\lambda}\exp(-\varepsilon X)\left(\log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \lambda X\right),$$

$$Lag_{25} = \frac{\varepsilon\exp(-\varepsilon t - \lambda X)}{2\lambda\left(\varepsilon t - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)},$$

$$Lag_{34} = -\frac{1}{2\varepsilon}\exp(-\varepsilon t + 2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2,$$

$$Lag_{36} = \frac{1}{\lambda^2}\exp(-\varepsilon t + \lambda X)\left(\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t} - \varepsilon\right)\log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),$$

$$Lag_{37} = \frac{1}{\varepsilon}\exp(\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\varepsilon}{\lambda}\right),$$

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$Lag_{17}, Lag_{25}, Lag_{34}$ admit five Noether symmetries

Lag_{68} (Volterra's Lagrangian) admits two Noether symmetries only.

Conservation laws

For example the Lagrangian *Lag_{34}* yields the following five Noether symmetries and corresponding first integrals of equation (13)

$$
\begin{aligned}
\Gamma_3 \quad &\Longrightarrow \quad Int_3 = \exp(\lambda X)\left(-\varepsilon + \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
\Gamma_4 \quad &\Longrightarrow \quad Int_4 = \exp(-\varepsilon t + \lambda X)\frac{\mathrm{d}X}{\mathrm{d}t},\\
\Gamma_5 \quad &\Longrightarrow \quad Int_5 = \exp(2\lambda X)\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2,\\
\Gamma_6 + 2\frac{\lambda}{\varepsilon}\Gamma_8 \quad &\Longrightarrow \quad Int_6 = \exp(-\varepsilon t + 2\lambda X)\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
\Gamma_7 \quad &\Longrightarrow \quad Int_7 = \exp(-2\varepsilon t + 2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2.
\end{aligned}
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[MCN and K.M.Tamizhmani, J. Nonlinear Math. Phys. 19 (2012)]

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- The physical Lagrangian admits the maximum number of Noether symmetries, i.e. FIVE.

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EMMY NOETHER

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ERWIN SCHRÖDINGER

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