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Willmore-Like Energies and Elastic Curves with Potential

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Introduction

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Extremals of $\mathcal{W}_\Phi$ are characterized by the equation

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- We are mainly interested in isometrically immersed surfaces on total spaces of Killing submersions.

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- They include all 3-dimensional homogeneous spaces with group of isometries of dimension 4.

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- In fact, all ξ-invariant surfaces of M can be seen as vertical lifts of curves.
- The mean curvature of these surfaces is **(Barros, 1997)**

$$H = \frac{1}{2}\left(\kappa \circ \pi\right) ,$$

κ denoting the geodesic curvature of γ in B.

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defined on the space of surface immersions in a total space of a Killing submersion with compact fibers, $Imm(N^2, M)$.

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Main Theorem (Barros, Garay & — , 2018)

If γ is a closed curve in B, then S_γ is a Willmore-like torus, if and only if, γ is an extremal of

$$\mathbf{\Theta}_{4\bar{\Phi}}(\gamma) = \int_\gamma \left(\kappa^2 + 4\bar{\Phi}\right)\, ds.$$

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Theorem **(Barros, Garay & — , 2018)**

A vertical torus S_γ is Willmore in M, if and only if, it is extremal of

$$\mathcal{W}_{\tau_\pi^2}(N^2) = \int_{N^2} \left(H^2 + \tau_\pi^2\right) \, dA.$$

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- **Corollary.** There exists a Killing submersion admitting a foliation by Willmore tori with CMC.

References

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THE END

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