Optimal control for discrete -time fractional order systems with Markovian jumps

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Optimal Control for Discrete- Time Fractional-Order Systems

- Fractional calculus(FC) began to engage mathematicians' interest in the 17th century as evidenced by a letter of Leibniz to L'Hôspital, dated 30^{th} September 1695, which talks about the possibility of non-integer order differentiation. Later on, famous mathematicians as Fourier, Euler and Laplace contributed to the foundation of this new branch of mathematics with various concepts and results.
- Nowadays, the most popular definitions of the non-integer order integral or derivative are the Riemann-Liouville, Caputo and Grunwald-Letnikov definitions. For a historical survey and the current state of the art, the reader is referred to [3], [4], [5],[6], [7] and the references therein.

Constructing the Grünwald–Letnikov derivative

The formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the derivative can be applied recursively to get higher-order derivatives. For example, the second-order derivative would be:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h_1 \to 0} \frac{\lim_{h_2 \to 0} \frac{f(x+h_1+h_2) - f(x+h_1)}{h_2} - \lim_{h_2 \to 0} \frac{f(x+h_2) - f(x)}{h_2}}{h_1}$$

Assuming that the h 's converge synchronously, this simplifies to:

$$= \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2},$$

which can be justified rigorously by the mean value theorem. In general, we have

$$f^{(n)}(x) = \lim_{h \to 0} \frac{\sum_{0 \le m \le n} (-1)^m \binom{n}{m} f(x + (n-m)h)}{h^n}.$$

Removing the restriction that n be a positive integer, it is reasonable to define:

$$\mathbb{D}^q f(x) = \lim_{h \to 0} \frac{1}{h^q} \sum_{0 \le m < \infty} (-1)^m \binom{q}{m} f(x + (q-m)h).$$

This defines the Grünwald–Letnikov derivative.

To simplify notation, we set:

$$\Delta_h^q f(x) = \sum_{0 \le m < \infty} (-1)^m \binom{q}{m} f(x + (q-m)h).$$

So the Grünwald–Letnikov derivative may be succinctly written as:

$$\mathbb{D}^q f(x) = \lim_{h \to 0} \frac{\Delta_h^q f(x)}{h^q}.$$

An alternative definition

In the preceding section, the general first principles equation for integer order derivatives was derived.

It can be shown that the equation may also be written as

$$f^{(n)}(x) = \lim_{h\to 0} \frac{(-1)^n}{h^n} \sum_{0\le m\le n} (-1)^m \binom{n}{m} f(x+mh).$$

or removing the restriction that n must be a positive integer:

$$\mathrm{D}^q f(x) = \lim_{h\to 0} \frac{(-1)^q}{h^q} \sum_{0\le m<\infty} (-1)^m \binom{q}{m} f(x+mh).$$

This equation is called the reverse Grünwald–Letnikov derivative. If the substitution $h \to -h$ is made, the resulting equation is called the direct Grünwald–Letnikov derivative:

$$\mathrm{D}^q f(x) = \lim_{h\to 0} \frac{1}{h^q} \boxed{\sum_{0\le m<\infty} (-1)^m \binom{q}{m} f(x-mh).}$$

The operator we shall use in the sequel

- FC finds use in different fields of science and engineering including the electrochemistry, electromagnetism, biophysics, quantum mechanics, radiation physics, statistics or control theory (see [6], [8], [4], [9]). Such an example comes from the field of autonomous guided vehicles, which lateral control seems to be improved by using fractional adaptation schemes [10]. Also, partial differential equations of fractional order were applied to model the wave propagation in viscoelastic media or the dissipation in seismology or in metallurgy [11].
- The optimal control theory was intensively developed during the last century for deterministic systems defined by integer-order derivatives, in both continuous- and discrete- time cases [12]. Since many real-world phenomena are affected by random factors that exercised a decisive influence on the processes behavior, stochastic optimal control theory had a similar evolution in the recent decades (see [13], [14], [15], [16] and the references therein). However, only a few papers address optimal control problems for fractional systems (see e.g. [17], [18], [19], [20], [21], [22]) and fewer consider stochastic fractional systems [23], [24].

Finite-horizon LQ optimal control problem

- In what follows we shall discuss a finite-horizon LQ optimal control problem for stochastic discrete-time LFSs defined by the Grunwald-Letnikov fractional derivative. As far as we know, this subject seems to be new for discrete-time LFSs with infinite Markovian jumps. The case of discrete-time LFSs affected by multiplicative, independent random perturbations was considered recently in *[*]Trujillo, J. J., & Ungureanu, V. M. (2018). Optimal control of discrete-time linear fractional-order systems with multiplicative noise. International Journal of Control, 91(1), 57-69]*. A determinstic case was considered previously in *A. Dzielinski,P. M. Czyronis, Dynamic Programming for Fractional Discrete-time Systems, 19th World Congress of IFAC,19(1)(2014)*.

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- Following [*], we have used an equivalent linear *expanded-state* model of the discrete-time LFS with jumps and we have rewritten the quadratic cost functional accordingly. Then the original optimal control problem reduces to a LQ optimal control problem for linear stochastic systems with Markovian jumps. The solution is obtained by

Why stochastic systems with Markovian jumps?

Stochastic discrete-time systems with Markovian switching can model many physical systems which may experience abrupt changes in their dynamics. Among them we mention the manufacturing systems, the power systems, the telecommunication systems e.t.c. All these systems suffer frequent unpredictable structural changes caused by failures or repairs, connections or disconnections of the subsystems [1].

Example

A model where the state vector x_k is computed by switching (according to a Markov law) between the following LFSs

$$\begin{aligned} State\ i &= 1 \\ \Delta_h^{[\alpha]} x_{k+1} &= \begin{pmatrix} -1/2 & 1 \\ 0 & -1/3 \end{pmatrix} x_k + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} u_k, k \in \mathbb{N} \\ State\ i &= 2 \\ \Delta_h^{[\alpha]} x_{k+1} &= \begin{pmatrix} -1/2 & 1 \\ 0 & -1/1 \end{pmatrix} x_k + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} u_k, k \in \mathbb{N} \\ x_0 &= x \in \mathbb{R}^d, \end{aligned}$$

Such a model could describe a special solar thermal receiver system where the state space $\mathcal{Z} = \{1, 2\}$ could represent an atmospheric condition where $i = 1$ means "sunny", $i = 2$ means "cloudy".

Stochastic systems with Markovian jumps

- $\{r_k\}_{k\in\mathbf{N}}$ is a homogeneous Markov chain on a probability space $(\Omega, \mathcal{F}, P)$, with the state space $\mathcal{Z} \subset \mathbb{Z}$ (finite or infinite) and $\mathcal{G}_k$ is the $\sigma-$ algebra generated by $\{r_i, 0 \leq i \leq k-1\}, k \in \mathbb{N}^*$. Consider the LFS with control

$$\begin{aligned} \Delta^{[\alpha]} x_{k+1} &= \mathbb{A}_k(r_k)x_k + \mathbb{B}_k(r_k)u_k, k \in \mathbb{N} && \text{(FOS)} \\ x_0 &= x \in \mathbb{R}^d, && \text{(Init cond FOS)} \\ y_{k+1} &= C_k(r_k)\, x_k && \text{(observable output)} \end{aligned}$$

where the control sequence $u = \{u_k\}_{k\in\mathbb{N}}$ (the input) belongs to a class of admissible controls $\mathcal{U}^a$ ($\mathcal{U}^a = \{u = \{u_k\}_{k\in\mathbb{N}}$ $| u_k \in L^2(\Omega, \mathbb{R}^m)$ is $\mathcal{G}_k$-measurable for all $k \in \mathbb{N}\}$) and the operator

$$\Delta^{[\alpha]} x_{k+1} = \frac{1}{h^\alpha} \sum_{j=0}^{k+1} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} x_{k+1-j}, h > 0$$

is the one used for the definition of the Grünwald-Letnikov fractional order derivative. Here $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$.

Optimal control problem

- Let $x \in \mathbb{R}^d$, $i \in \mathcal{Z}$ and $N \in \mathbb{N}$ be fixed. Our *optimal control problem* $\mathcal{O}$ is to minimize the cost functional

$$I_{x,N,i}(u) = \qquad \text{(cost funct)}$$

$$\sum_{n=0}^{N-1} E\left[\left(\left\|C_n(r_n)x_n\right\|^2 + < K_n(r_n)u_n, u_n >\right)|_{r_0=i}\right] +$$

$$E\left[< Sx_N, x_N > |_{r_0=i}\right]$$

subject to (FOS)-(Init cond FOS), over the class $\mathcal{U}^a$ of admissible controls.

- Here $S \geq \delta I_{\mathbb{R}^m}$, $K_n(i) \geq \delta_n I_{\mathbb{R}^m}$,for all $i \in \mathcal{Z}$ and $E[\xi|\eta = x]$ denotes the conditional expectation on the event $\eta = x$ of an integrable, real-valued random variable ξ defined on a probability space $(\Omega, \mathcal{F}, P)$.

A linear expanded-state model- The coefficients

For all $j \in \mathbb{N}$, $c_j := (-1)^j \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$, $A_k^0(i) = h^\alpha \mathbb{A}_k(i) + \alpha I_{\mathbb{R}^d}$, $B_k(i) = h^\alpha \mathbb{B}_k(i)$,$i \in \mathcal{Z}$ and $\mathcal{A}_k(i) \in L\left(\left(\mathbb{R}^d\right)^N\right)$, $\mathcal{B}_k(i) \in L\left(\mathbb{R}^m, \left(\mathbb{R}^d\right)^N\right)$, $k \in \mathbb{N}$ are linear and bounded operators defined by

$$\mathcal{A}_k(i) = \begin{pmatrix} A_k^0(i) & c_1 I_{\mathbb{R}^d} & . & c_{N-1} I_{\mathbb{R}^d} \\ I_{\mathbb{R}^d} & 0 & . & 0 \\ . & I_{\mathbb{R}^d} & . & . \\ . & . & . & 0 \\ 0 & . & I_{\mathbb{R}^d} & 0 \end{pmatrix}, \mathcal{B}_k(i) = \begin{pmatrix} B_k(i) \\ 0 \\ . \\ 0 \end{pmatrix}. \tag{1}$$

Also,

$$\mathcal{C}_k(i) : \left(\mathbb{R}^d\right)^N \to \mathbb{R}^p, \mathcal{C}_k(i)(v_0, v_1, ...) = C_k(i)(v_0), k = 0, .., N-1$$
$$\mathcal{S} : (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N, \mathcal{S}(v_0, ..., v_{N-1}) = (Sv_0, 0, ..., 0).$$

A linear expanded-state model-The equations

$$X_k^T = \left(x_k, x_{k-1}, ..., x_0, 0, .., \underset{N}{0} \right) \text{ and}$$

$$\begin{aligned} X_{k+1} &= \mathcal{A}_k(r_k) X_k + \mathcal{B}_k(r_k) u_k, & (2) \\ X_0 &= (x_0, 0, 0...) \in l^2_{\mathbb{R}^d}, x_0 = x \in \mathbb{R}^d & (3) \end{aligned}$$

$$\begin{aligned} I_{x,N,i}(u) = \sum_{n=0}^{N-1} E[(\langle \mathcal{C}_k^*(r_k) \mathcal{C}_k(r_k) X_k, X_k \rangle + \langle K_k(r_k) u_k, u_k \rangle) |_{r_0=i}] \\ +E[(\langle (\mathcal{A}_{N-1}^*(r_{N-1}) \mathcal{S} \mathcal{A}_{N-1}(r_{N-1})) X_{N-1}, X_{N-1} \rangle \\ +2 \langle (\mathcal{A}_{N-1}^*(r_{N-1}) \mathcal{S} \mathcal{B}_{N-1}(r_{N-1})) u_{N-1}, X_{N-1} \rangle + \\ \langle (\mathcal{B}_{N-1}^*(r_{N-1}) \mathcal{S} \mathcal{B}_{N-1}(r_{N-1})) u_{N-1}, u_{N-1} \rangle)|_{r_0=i}]. \qquad (4) \end{aligned}$$

Backward discrete-time Riccati equation of control on ordered Banach spaces

- H is a Hilber space, $L(H)$ denotes the Banach space of all linear and bounded operators $A : H \to H$ and $\mathbb{S}(H)$ is the Banach subspace of $L(H)$ formed by all self-adjoint operators. $l^{\mathcal{Z}}_{\mathbb{S}(\mathbb{R}^d)} = \{R = \{R(i) \in \mathbb{S}(H), i \in \mathcal{Z}\}, \|R\|_{\mathcal{Z}} = \sup_{i\in\mathcal{Z}} \|R(i)\| < \infty\}$ -Banach orderd space
- $\mathfrak{E} : l^{\mathcal{Z}}_{\mathbb{S}(\mathbb{R}^d)} \to l^{\mathcal{Z}}_{\mathbb{S}(\mathbb{R}^d)}, \mathfrak{E}(R)(i) = \sum_{j\in\mathcal{Z}} q_{ij} R(j), q_{ij} =$ $P(r_{n+1} = j|_{r_n=i})$(transition matrix of the Markov chain)
- for all $i \in \mathcal{Z}, k \in \{0, 1, .., N-1\}$

$$
\begin{aligned}
P_k(i) &= \mathcal{A}_k^*(i)\,\mathfrak{E}(P_{k+1})(i)\,\mathcal{A}_k(i) + \mathcal{C}_k^*(i)\,\mathcal{C}_k(i) \qquad \text{(Riccati Syst)}\\
-&[\mathcal{A}_k^*(i)\,\mathfrak{E}(P_{k+1})(i)\,\mathcal{B}_k(i)]\cdot[K_k(i) + \mathcal{B}_k^*(i)\,\mathfrak{E}(P_{k+1})(i)\,\mathcal{B}_k(i)]^{-1}\cdot\\
&[\mathcal{B}_k^*(i)\,\mathfrak{E}(P_{k+1})(i)\,\mathcal{A}_k(i)],\\
P_N(i) &= \mathcal{S} \qquad \text{(Final cond)}
\end{aligned}
$$

Cost functional

Theorem

The cost functional (4) can be equivalently rewritten as

$$I_{x,N,i}(u) = E\left[\langle P_0(i) X_0, X_0\rangle |_{r_0=i}\right] + \tag{5}$$
$$+\sum_{n=0}^{N-1} E\left[\left\| \left[\mathcal{K}_n(P_{n+1})(r_n)\right]^{1/2}\left[W_n(r_n) X_n - u_n\right]\right\|^2 |_{r_0=i}\right],$$

where, for all $R \in l^{\mathcal{Z}}_{\mathbb{S}(\mathbb{R}^d)}, i \in \mathcal{Z},\ n \in \{0,1,..,N-1\}$,

$$\begin{aligned}\mathcal{K}_n(R)(i) &= K_n(i) + \mathcal{B}_n^*(i)\,\mathfrak{E}(R)(i)\,\mathcal{B}_n(i),\\ W_n(i) &= -\left[\mathcal{K}_n(P_{n+1})(i)\right]^{-1}\cdot\left[\mathcal{B}_n^*(i)\,\mathfrak{E}(P_{n+1})(i)\,\mathcal{A}_n(i)\right].\end{aligned}$$

Main result

Theorem

Let $\{P_n\}_{n=0,..,N-1}$ be the unique solution of the generalized Riccati equation of control and let W_n, $n = 0, .., N-1$ be defined as above. The control sequence

$$\widetilde{u} = \{\widetilde{u}_0 = W_0 X_0, ..., \widetilde{u}_n = W_n X_n, ..., \widetilde{u}_{N-1} = W_{N-1} X_{N-1}\} \tag{6}$$

minimizes the cost functional $I_{x,N,i}(u)$ and the optimal cost is

$$\min_{u \in \mathcal{U}^a_{0,N-1}} I_{x,N,i}(u) = E\left[\langle P_0 X_0, X_0 \rangle \,|_{r_0 = i}\right]. \tag{7}$$

Example

Consider a time-invariant, finite-dimensional version of the LFS with control where $\alpha = \frac{1}{2}$, $h = 2$, $d = 2$, $m = 2$, $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathcal{Z} = \{1, 2\}$ and

$\mathbb{A}_k(1) = \begin{pmatrix} -1/2 & 1 \\ 0 & -1/3 \end{pmatrix}$, $\mathbb{A}_k(2) = \begin{pmatrix} -1/2 & 1 \\ 0 & -1/1 \end{pmatrix}$,

$Q = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$(Transition matrix of the markov process),$K_k(1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

$K_k(2) = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and

$B_k(i) = \begin{pmatrix} 1/(i+1) & 0 \\ 0 & 1/(i+1) \end{pmatrix}$,

$C_k(i) = (i+1)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, for $i = 1, 2$.

Example

In this example, the state space of the Markon chain is $\mathcal{Z} = \{1, 2\}$ and the state vector x_k will be computed by switching (according to the Markov law) between the following two deterministic LFSs

$$\begin{aligned} \Delta^{[\alpha]} x_{k+1} &= \begin{pmatrix} -1/2 & 1 \\ 0 & -1/3 \end{pmatrix} x_k + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} u_k, k \in \mathbb{N} \\ x_0 &= x \in \mathbb{R}^d, \end{aligned}$$

$$\begin{aligned} \Delta^{[\alpha]} x_{k+1} &= \begin{pmatrix} -1/2 & 1 \\ 0 & -1/1 \end{pmatrix} x_k + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} u_k, k \in \mathbb{N} \\ x_0 &= x \in \mathbb{R}^d, \end{aligned}$$

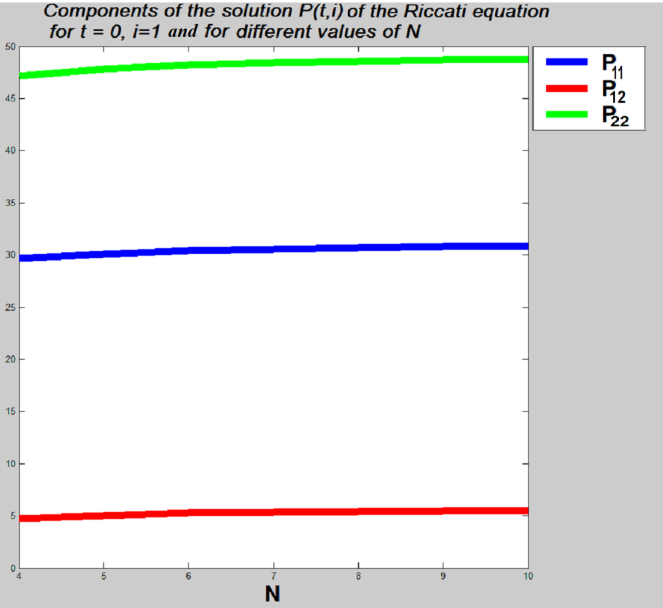
Our model could describe a special solar thermal receiver system where the state space $\mathcal{Z} = \{1, 2\}$ of the Markov chain could represent an atmospheric condition where $i = 1$ means "sunny", $i = 2$ means "cloudy".

Example

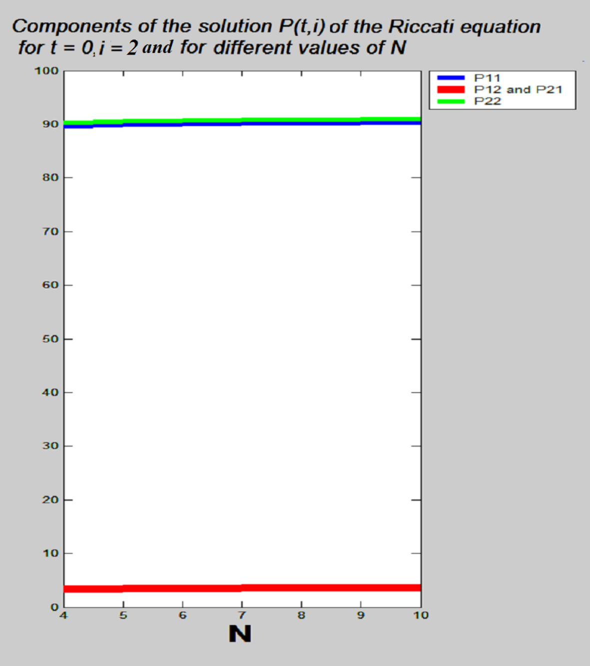
$P_0(i)$ is a defined by a $(2 \times N) \times (2 \times N)$ symmetric matrix. The relevant values of $P_0(i)$ correspond to the first 2 nonzero elements of $X_0 = \left(x_0, 0, .., \underset{N-1}{0}\right), x_0 \in \mathbb{R}^2$ because $\min_{u \in \mathcal{U}^a_{0,N-1}} I_{x,N,i}(u) = E\left[\langle P_0 X_0, X_0 \rangle |_{r_0=i}\right].$

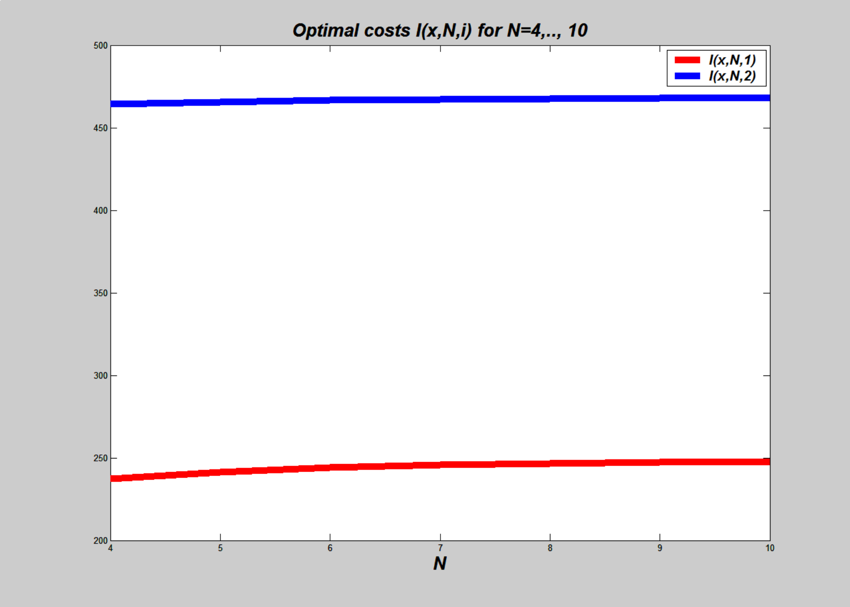
$$P_0(i) \overset{not}{=} P(0,i) = \begin{pmatrix} P_{11}(0,i) & P_{12}(0,i) & . & . & P_{1(2N)}(0,i) \\ P_{12}(0,i) & P_{22}(0,i) & . & . & P_{2(2N)}(0,i) \\ . & . & . & . & . \\ . & . & & & \\ . & . & & & \\ . & . & & & \\ P_{1(2N)}(0,i) & P_{2(2N)}(0,i) & & & \end{pmatrix}$$

By implementing in Matlab the above results concerning the Riccati equation of control and the optimal cost, we get the following numerical results.



Components of the solution P(t,i) of the Riccati equation
for t = 0, i=1 and for different values of N
50
45
40
35
30
25
20
15
10
5
0
4
5
6
7
8
9
10
N
P_{11}
P_{12}
P_{22}


Components of the solution P(t,i) of the Riccati equation
for t = 0, i = 2 and for different values of N
P11
P12 and P21
P22
100
90
80
70
60
50
40
30
20
10
0
4
5
6
7
8
9
10
N


Optimal costs I(x,N,i) for N=4,.., 10
I(x,N,1)
I(x,N,2)
500
450
400
350
300
250
200
4
5
6
7
8
9
10
N

Conclusions and further research

This paper provides a new method (based on the dynamic programming approach) for solving the LQ optimal control problem $\mathcal{O}$. It consists in a reformulation of the problem for an associated linear non-fractional system (2)-(3), defined on spaces of higher dimensions. The computer program implementing this method is simple and fast.The linear approach opens the way of solving an infinte horizon LQ optimal control problem for LFSs.

Further research

- infinite horizon LQ optimal control problem for LFSs
- asympototic behavior of the generalized Riccati equation with control
- stabilizability, detectability and observability properties of the LFS (other than the exponential stability, because the system cannot be exponentially stabilized).

Thank you!!

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