

# Contents

<b>Introduction</b>	<b>i</b>
<b>List of notation</b>	<b>vii</b>
<b>1 Geometry and variational calculus</b>	<b>1</b>
1.1 Planar curves	1
1.1.1 Polar coordinate system	2
1.1.2 Curvature	3
1.1.3 Frenet-Serret equations	5
1.2 Space curves	6
1.2.1 Unit tangent vector	6
1.2.2 Tangent line and normal plane	8
1.2.3 Curvature	9
1.2.4 Principal normal vector	13
1.2.5 Principal and oscular plane	14
1.2.6 Binormal and moving trihedron	16
1.2.7 Torsion	18
1.2.8 Frenet-Serret equations of space curves	23
1.2.9 Main theorem in the local theory of curves	25
1.3 Surfaces	26
1.3.1 Surfaces and parametric curves	26
1.3.2 First fundamental form	27
1.3.3 Normal vector to the surface	28
1.3.4 A second fundamental form	29
1.3.5 Integrability conditions	33
1.4 Variational calculus	36
1.4.1 Euler-Lagrange equation	36
1.4.2 First integrals of the Euler-Lagrange equation	38
1.4.3 Euler-Lagrange equation for two independent variables	44
1.4.4 Euler-Lagrange equation for Lagrangians containing second order derivatives	48

<b>2 Planar curves which curvature depends only on the distance from a fixed point</b>	<b>49</b>
2.1 The moving frame associated with a plane curve	49
2.2 Integration	51
2.3 Bernoulli's lemniscates	51
2.4 Relationship between the lemniscate and the elastica	54
2.5 Spirals	55
2.6 Sturm spirals	56
2.7 Generalized Sturm spirals	57
2.7.1 The case when $\sigma > 1$	57
2.7.2 The case when $0 < \sigma < 1$	59
2.7.3 The sub-case when $0 < \sigma < 1$ and $c = 0$	60
2.8 Serret curves	61
2.8.1 Generalized Serret curves	63
2.9 Cassinian ovals	65
2.9.1 Alternative parameterizations	66
<b>3 Biological membranes</b>	<b>71</b>
3.1 Subject matter and biological membranes	71
3.2 Types of membranes	73
3.3 Functions of biomembranes	74
3.4 Chemical composition and physical properties of biomembranes	75
3.4.1 Molecular structure and physicochemical properties of membrane lipids	76
3.4.2 Membrane proteins and glycoproteins	78
3.5 Membrane models and methods for the study of biomembranes	80
3.5.1 Modern theories	80
3.6 Model membrane structures	82
3.6.1 Lipid associates	82
3.6.2 Model artificial membranes	85
<b>4 Surface tension and balance</b>	<b>89</b>
4.1 Mechanical equilibrium	89
4.1.1 Laplace – Young equation	89
4.1.2 Axially symmetric membranes	90
4.1.3 Stresses	92

4.1.4 The case $w = 0$	94
4.1.5 Forms and the corresponding surfaces	97
4.2 Delaunay surfaces	97
4.2.1 Nodoids and unduloids	98
4.2.2 Intrinsic equation of the profile curves of Delaunay surfaces	100
4.2.3 Some useful formulas	102
4.2.4 Delaunay construction	103
4.2.5 Nodary	105
4.2.6 Undulary	106
4.3 Polyester balloon and elastic curves	109
4.3.1 Bending energy	112
4.3.2 Original formulation and treatment of the problem about elastic curves	113
4.3.3 Parametric representation of curvature of Elastica	116
4.3.4 Intrinsic equation of the Elastica	117
4.3.5 Form of a hanging chain	119
4.3.6 One-dimensional membranes – Euler's elasticas	122
4.4 Whewell parameterization	124
4.4.1 Introduction	124
4.4.2 Equilibrium equations	125
4.4.3 Elastics with tension	128
4.5 Geometry of the rotating liquid drop	132
4.5.1 Surface invariants	133
4.5.2 Parameterization by Legendrian integrals	135
4.5.3 Parameterization by Weierstrassian functions	135
4.5.4 Intrinsic equation of the profile curves	138
4.5.5 Geodesic curves	138
4.5.6 Non-resolved issues	140
<b>5 Equations of equilibrium states of membranes</b>	<b>143</b>
5.1 Canham model	143
5.1.1 Key assumptions in the model	144
5.2 Helfrich and Deuling model	145
5.3 Model of Ou-Yang and Helfrich	147
5.3.1 Basic formulas and definitions	148

5.3.2 Equation of the form	151
5.4 Symmetries of the form equation	152
5.4.1 Cartesian coordinates	153
5.4.2 Group-invariant solutions	154
5.4.3 Conformal coordinates	155
5.4.4 Lie equations	157
5.4.5 Determining system of equations	158
<b>6 Exact solutions and applications</b>	<b>161</b>
6.1 Unduloids and nerve fibers	161
6.1.1 Introduction	161
6.1.2 Model	163
6.1.3 Parameterization	163
6.1.4 Parameters of the nerve fibers	165
6.1.5 Sensitivity of the equilibrium forms on the parameters	167
6.2 Mathematical model of the Cole experiment	169
6.2.1 Cole model	170
6.2.2 Yoneda method	172
6.2.3 Nodoids and the compression of the spherical eggs	173
6.3 Fusion of membranes	176
6.3.1 Stalk model	176
6.3.2 Mathematics of the stalk model	177
6.3.3 Geometric and energetic aspects	180
6.4 Cylindrical membranes	182
6.4.1 Translational-invariant solutions	182
6.4.2 Analytical solutions	184
6.4.3 Closure conditions	186
6.4.4 Self-intersections	188
6.4.5 Hele-Shaw cells	189
6.5 Beyond Delaunay's surfaces	190
6.5.1 Parametric equations	192

<b>Epilogue</b>	<b>197</b>
<b>Appendix A</b>	<b>199</b>
A.1 Elliptic integrals and functions	199
A.2 Jacobian elliptic functions	200
A.3 Weierstrassian elliptic functions	204
<b>Bibliography</b>	<b>207</b>