Chapter 2
Planar Curves Whose Curvature Depends Only on the Distance From a Fixed Point

Abstract Looking at the Frenet-Serret equations from the viewpoint of dynamical systems, one can prove that when the curvature of a plane curve is given as a function of the radius, the problem of reconstructing this curve is reducible to quadratures. Additionally, two different integration procedures are presented. These methods are illustrated first via the famous lemniscate of Bernoulli, which is immediately related to the Euler elastica. Relying on the new formalism, the Sturm spirals and their generalizations, the Serret curves (which have a mechanical origin) and their generalizations are parametrized explicitly. The results on the Serret curves are original, as their description up to now has been purely abstract. Finally, the same technique is applied to the Cassinian ovals, and in this way, one concludes with their alternative parameterizations.

2.1 The Moving Frame Associated with a Plane Curve

The fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see Berger and Gostiaux (1988), p. 296 or Oprea (2007), p. 37). The simplicity of the situation, however, is elusive, as in many cases, it is impossible to find the curve explicitly. Having that in mind, it is clear that if the curvature is given as a function of its position, the situation is even more complicated. Viewing the Frenet-Serret equations as a fictitious dynamical system, it was proven in Vassilev et al. (2009) (see also Djondjorov et al. (2009a)) that when the curvature is given simply as a function of the distance from the origin, the problem can always be reduced to quadratures. This last result is not entirely new, as Singer (1999) had already shown that in some cases, it is possible that such a curvature has an interpretation as a central potential in the plane, and therefore the trajectories can be found through the standard procedures in classical mechanics. However, the approach which we will follow here is entirely different from the group-theoretical approach of Vassilev et al. (2009) or the mechanical approach of Singer (1999) proposed in those papers. The method is illustrated by the most natural example in the class of curves whose curvatures are functions only of the distance from the

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