Chapter 4
Surface Tension and Equilibrium

Abstract In this Chapter, the equilibria of membranes is treated from the perspective of the first principles of mechanics. For example, the Laplace–Young equation, which is valid for an arbitrary infinitesimal area of the membrane, is derived in full detail just to clarify the appearance of membrane curvatures in all further considerations. If we limit ourselves to the class of axially symmetric membranes, the corresponding balance equations can be written in a vector form, which is valid along the whole surface. Projection of this vector equation along the normal and tangential directions at any point leads to a system of two coupled nonlinear equations. It turns out that this system can be solved in two cases of direct interest. Namely, Delaunay surfaces and the polyester balloon, both of which are described explicitly. It should also be noted that the so-obtained parameterization of Delaunay surfaces is regular, contrary to the one which follows from the original construction. Here, we also consider the problem of a fluid body rotating with a constant angular velocity and subjected to surface tension of ambient interface. Determining the equilibrium configuration of this system turns out to be equivalent to the geometrical problem of finding the surface of revolution with a prescribed mean curvature. In the simply connected case, the equilibrium surface can be parameterized explicitly via elliptic integrals of the first and second kinds. Here, we present two different parameterizations of rotating drops based on the Jacobian and Weierstrassian elliptic functions and integrals. By making use of the second of these, we are able to study the finer details of the drop surfaces, such as the existence of closed geodesics and other quantities of geometrical and mechanical interest.

4.1 Mechanical Equilibrium

4.1.1 Laplace–Young Equation

Let us consider the infinitesimal curvilinear quadrangle *ABCD* of the membrane. If the point *A* coincides with the origin of the orthogonal coordinate lines on the membrane with $\mathcal{R}_u$ and $\mathcal{R}_v$ being their curvature radii and $p_{\rm in}, p_{\rm out}$ being, respectively, the inner and outer pressures, the work W needed for the infinitesimal expansion of

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