

Contents

1	Geometry and Variational Calculus	1
1.1	Plane Curves	1
1.1.1	Polar Coordinate System	2
1.1.2	Curvature	3
1.1.3	Frenet-Serret Equations	5
1.2	Space Curves	6
1.2.1	Unit Tangent Vector	6
1.2.2	Tangent Line and Normal Plane	8
1.2.3	Curvature	9
1.2.4	Principal Normal Vector	13
1.2.5	Principal and Osculating Plane	13
1.2.6	Binormal and Moving Frame	15
1.2.7	Torsion	17
1.2.8	Frenet-Serret Equations of Space Curves	22
1.2.9	Main Theorem in the Local Theory of Curves	23
1.3	Surfaces	25
1.3.1	Surfaces and Parametric Curves	25
1.3.2	First Fundamental Form	26
1.3.3	Normal Vector to the Surface	26
1.3.4	A Second Fundamental Form	27
1.3.5	Integrability Conditions	31
1.4	Variational Calculus	33
1.4.1	Euler-Lagrange Equation	33
1.4.2	First Integrals of the Euler-Lagrange Equation	36
1.4.3	Euler-Lagrange Equation for Two Independent Variables	41
1.4.4	Euler-Lagrange Equation for Lagrangians Containing Second Order Derivatives	45
	References	46

2 Planar Curves Whose Curvature Depends Only on the Distance From a Fixed Point	47
2.1 The Moving Frame Associated with a Plane Curve	47
2.2 Integration	48
2.3 Bernoulli's Lemniscates	49
2.4 Relationship Between the Lemniscate and the Elastica	52
2.5 Spirals	52
2.6 Sturmian Spirals	53
2.7 Generalized Sturm Spirals.	54
2.7.1 The Case When $\sigma > 1$	54
2.7.2 The Case When $0 < \sigma < 1$	56
2.7.3 The Sub-case When $0 < \sigma < 1$ and $c = 0$	57
2.8 Serret Curves	59
2.8.1 Generalized Serret Curves.	60
2.9 Cassinian Ovals	62
2.9.1 Alternative Parameterizations	63
References.	67
3 Biological Membranes	69
3.1 Subject Matter and Biological Membranes	69
3.2 Types of Membranes	70
3.3 Functions of Biomembranes	71
3.4 Chemical Composition and Physical Properties of Biomembranes	72
3.4.1 Molecular Structure and Physicochemical Properties of Membrane Lipids	73
3.4.2 Membrane Proteins and Glycoproteins	74
3.5 Membrane Models and Methods for the Study of Biomembranes	76
3.5.1 Modern Theories.	76
3.6 Model Membrane Structures.	77
3.6.1 Lipid Associates	77
3.6.2 Model Artificial Membranes	80
References.	82
4 Surface Tension and Equilibrium	83
4.1 Mechanical Equilibrium	83
4.1.1 Laplace–Young Equation	83
4.2 Tensions and Geometry	84
4.2.1 Membrane Geometry.	84
4.2.2 Tensions	85
4.2.3 The Case $w = 0$	87
4.2.4 Shapes and the Corresponding Surfaces	89

4.3	Delaunay Surfaces	89
4.3.1	Nodoids and Unduloids.	90
4.3.2	Intrinsic Equation of the Profile Curves of Delaunay Surfaces	92
4.3.3	Some Useful Formulas	94
4.3.4	Delaunay Construction	95
4.3.5	Nodary	96
4.3.6	Undulary	98
4.4	Mylar Balloon and Elastic Curves	100
4.4.1	Bending Energy	103
4.4.2	Original Formulation and Treatment of the Problem About Elastic Curves	104
4.4.3	Parametric Representation of the Curvature of Elastica	108
4.4.4	Intrinsic Equation of the Elastica.	109
4.4.5	A Hanging Chain	111
4.4.6	Eulers Elasticas as One-Dimensional Membranes	113
4.5	Whewell Parameterization.	115
4.5.1	Equilibrium Equations.	116
4.5.2	Elasticas With Tension	119
4.6	Elastic Sturmian Spirals	122
4.6.1	Explicit Analytical Solutions.	123
4.7	Alternative Parameterization of Elastic Spirals	127
4.7.1	Main Calculations	127
4.8	Geometry of the Rotating Liquid Drop.	134
4.8.1	Introduction	134
4.8.2	Geometry and Surface Invariants	135
4.8.3	Parameterization via Legendre's Integrals	138
4.8.4	Parameterization via Weierstrass's Functions	139
4.8.5	Geodesics on the Drop.	141
4.8.6	Questions for Future Work	145
	References	147
5	Equations of Equilibrium States of Membranes	151
5.1	Canham Model	151
5.2	Key Assumptions in the Model	152
5.3	Helfrich and Deuling Model	153
5.4	Ou-Yang and Helfrich Model	156
5.4.1	Basic Formulas and Definitions.	156
5.4.2	Shape Equation	159
5.5	Symmetries of the Shape Equation	160
5.5.1	Cartesian Coordinates	160
5.5.2	Group-Invariant Solutions	162
	5.5.3 Conformal Coordinates	163

5.5.4 Lie Equations	164
5.5.5 Determining System of Equation.	165
References.	166
6 Exact Solutions and Applications.	167
6.1 Unduloids and Nerve Fibers.	168
6.1.1 Introduction.	168
6.1.2 Model	169
6.1.3 Parameterization	169
6.1.4 Parameters of the Nerve Fibers	171
6.1.5 Sensitivity of the Equilibrium Shapes on the Parameters	173
6.2 Mathematical Model of the Cole Experiment	175
6.2.1 Cole Model	176
6.2.2 Yoneda Method.	178
6.2.3 Nodoids and the Compression of Spherical Eggs	178
6.3 Fusion of Membranes.	181
6.3.1 Stalk Model.	181
6.3.2 Mathematical Description of the Stalk Model	182
6.3.3 Geometric and Energetic Aspects	184
6.4 Cylindrical Membranes.	186
6.4.1 Translational-Invariant Solutions	186
6.4.2 Analytical Solutions	187
6.4.3 Closure Conditions	190
6.4.4 Self-Intersections.	191
6.4.5 Hele-Shaw Cells	192
6.5 Beyond Delaunay Surfaces.	194
6.5.1 Parametric Equations.	195
References.	198
Epilogue.	201
Appendix A: Elliptical Integrals and Functions	203
Index	211