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QUARTER TURNS AND NEW FACTORIZATIONS OF ROTATIONS

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Abstract

Here we consider new decompositions of the special orthogonal transformations in $\mathbb{R}^3$ into products of two rotations, one of them has a fixed scalar parameter, and the other – a fixed axis. The obtained analytic solutions constitute an alternative parametrization of the group $\mathsf{SO}(3)$ with charts in $\mathbb{S}^2 \times \mathbb{S}^1$. As it should be expected, from topological point of view, this map has singularities – the number of images varies between zero, one, two and infinitely many. The corresponding formulae become particularly simple in the cases involving quarter turns and half turns, although in the latter additional geometric criteria appear. Transferring the same construction to the universal cover $\mathsf{SU}(2) \cong \mathbb{S}^3$ via quaternion parametrization eliminates the problem with infinite scalar parameters. The so obtained map can also be seen as a realization of the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$.

Key words: decomposition, half turn, quarter turn, quaternion, vector-parameter

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1. Introduction. It is well known (Euler's theorem) that all three dimensional rotations have an invariant axis which is unique except in the trivial case (identity transformation) and can be specified by a unit vector $\hat{\mathbf{n}}$. In order to describe the rotation itself, we need an extra parameter, usually chosen to be the angular variable φ. Then we may construct the matrix of the transformation using the famous Rodrigues' formula

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) = \cos\varphi\, \mathcal{I} + (1 - \cos\varphi)\, \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t + \sin\varphi\, \hat{\mathbf{n}}^\times, \tag{1}$$

where $\mathcal{I}$ is the identity matrix in $\mathbb{R}^3$, $\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}^t$ is a projector along $\hat{\mathbf{n}}$ obtained by the usual *tensor* or *dyadic* product of vectors and $\hat{\mathbf{n}}^\times$ denotes the skew-symmetric matrix associated with $\hat{\mathbf{n}}$ via the Hodge duality

$$\mathbb{R}^3 \ni \mathbf{c} \to \mathbf{c}^\times, \qquad \left(\mathbf{c}^\times\right)_{ij} = \varepsilon_{ilj} c_l \ \Rightarrow\ \mathbf{c}^\times \boldsymbol{\xi} = \mathbf{c} \times \boldsymbol{\xi}, \qquad \boldsymbol{\xi} \in \mathbb{R}^3.$$

In what follows we use special notations in the particular cases of half turns (rotations by a straight angle):

$$\mathcal{O}(\hat{\mathbf{n}}) = \mathcal{R}(\hat{\mathbf{n}}, \pi) = 2\,\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t - \mathcal{I} \tag{2}$$

and similarly for quarter turns (rotations by a right angle) which have the form

$$\mathcal{Q}(\hat{\mathbf{n}}) = \mathcal{R}(\hat{\mathbf{n}}, \pi/2) = \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t + \hat{\mathbf{n}}^\times. \tag{3}$$

Moreover, we could obtain another representation of $\mathcal{R}(\hat{\mathbf{n}}, \varphi)$ using the well known trigonometric Euler substitution $\tau = \tan\dfrac{\varphi}{2}$. With its help equation (1) becomes

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) = \frac{(1-\tau^2)\,\mathcal{I} + 2\tau^2\,\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t + 2\tau\,\hat{\mathbf{n}}^\times}{1+\tau^2}$$

and once the *vector parameter* of the rotation is defined to be $\mathbf{c} = \tau\hat{\mathbf{n}}$, the representation

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c} \otimes \mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2} \tag{4}$$

is straightforward.

Note that we no longer need to write two arguments – the angle is parameterized by τ, which we refer to as *scalar parameter* and it is included in the definition of $\mathbf{c}$.

There is a natural composition law for the vector parameters that comes quite handy for our considerations. Namely, from the group composition in $\mathsf{SO}(3)$, it can be derived (see [3,7,9]) that the vector parameter of the composition $\mathcal{R}(\mathbf{a})\,\mathcal{R}(\mathbf{b}) = \mathcal{R}(\langle\mathbf{a}, \mathbf{b}\rangle) = \mathcal{R}(\mathbf{c})$ is

$$\mathbf{c} = \langle\mathbf{a}, \mathbf{b}\rangle = \frac{\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b}}{1 - (\mathbf{a}, \mathbf{b})}\,\cdot \tag{5}$$

We refer to [3] for details regarding the derivation of (5) and to [7,8] for some mechanical applications of the vector parametrization, while references [1,10] can be consulted for a thorough discussion of Euler's invariant axis theorem and Rodrigues' formula (1).

2. Generic construction. In this section we consider the generic case in which an Euclidean rotation $\mathcal{R}(\mathbf{c})$ can be decomposed into a pair of rotations

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(v\hat{\mathbf{c}}_2)\mathcal{R}(u\hat{\mathbf{c}}_1) \tag{6}$$

for one of which we know to be the axis, determined by the unit vector $\hat{\mathbf{c}}_1$ or $\hat{\mathbf{c}}_2$ and for the other – the scalar parameter v or u respectively.

Case 1. If v and $\hat{\mathbf{c}}_1$ are given and u, $\hat{\mathbf{c}}_2$ are unknown, we multiply with $\mathcal{R}(-u\hat{\mathbf{c}}_1) = \mathcal{R}^{-1}(u\hat{\mathbf{c}}_1)$ on the right and it is straightforward to express $\hat{\mathbf{c}}_2$ by means of (5)

$$v\hat{\mathbf{c}}_2 = \langle \mathbf{c}, -u\hat{\mathbf{c}}_1 \rangle = \frac{\tau\hat{\mathbf{n}} - u\hat{\mathbf{c}}_1 + u\tau\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}}}{1 + u\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)} \cdot \tag{7}$$

An additional requirement should hold in order to guarantee the existence of such decomposition. Since $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ are eigenvectors of $\mathcal{R}(u\hat{\mathbf{c}}_1)$ and $\mathcal{R}(v\hat{\mathbf{c}}_2)$ respectively, with the notation $\sigma_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j)$ one has

$$\sigma_{21} = (\hat{\mathbf{c}}_2, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2, \hat{\mathbf{c}}_1) = \cos\gamma, \tag{8}$$

where γ is the angle between $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$.

On the other hand, considering the scalar product of (7) with $\hat{\mathbf{c}}_1$ one obtains

$$v\cos\gamma = \frac{\tau\rho_1 - u}{1 + u\tau\rho_1} \tag{9}$$

with $\rho_i = (\hat{\mathbf{n}}, \hat{\mathbf{c}}_i) = \cos\measuredangle(\hat{\mathbf{n}}, \hat{\mathbf{c}}_i)$.

In order to determine the scalar parameter u, it suffices to consider another matrix entry

$$\sigma_{11} = (\hat{\mathbf{c}}_1, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_1, \mathcal{R}(v\hat{\mathbf{c}}_2)\,\hat{\mathbf{c}}_1) = \cos\psi_2 + (1 - \cos\psi_2)\cos^2\gamma, \tag{10}$$

where $\psi_2 = 2\arctan(v)$ is the generalized Euler angle of rotation about $\hat{\mathbf{c}}_2$ and for the last equality we use the explicit form of $\mathcal{R}(v\hat{\mathbf{c}}_2)$, given in (1).

Now it remains to substitute (10) in (9) and express $\cos\psi_2$ in terms of $u = \tan\dfrac{\psi_2}{2}$ in order to obtain the two solutions for the scalar parameter u

$$u_\pm = \frac{\tau\rho_1 - v\cos\gamma_\pm}{1 + v\tau\rho_1\cos\gamma_\pm}, \qquad v\cos\gamma_\pm = \pm\frac{1}{\sqrt{2}}\sqrt{(v^2+1)\,\sigma_{11} + v^2 - 1}\,. \tag{11}$$

Note that the above formula gives rise to another condition: $\sigma_{11} \geq \cos\psi_2$ that guarantees real solutions, since only they correspond to rotations.

These two solutions, substituted in (7), determine the two unit vectors $\hat{\mathbf{c}}_2^{\pm}$, or the two possible axes for the second rotation in the decomposition.

Obviously, in the case $v = \tau$ one of these solutions is trivial: $u = 0$, $\hat{\mathbf{c}}_2 = \hat{\mathbf{n}}$ and if, on the other hand, $\cos\gamma = 0$ ($\sigma_{11} = \cos\psi_2$), we end up with a double root $u_+ = u_- = \tau\rho_1$. If $\rho_1 = 0$ in particular, it is not difficult to assure ourselves by a geometric argument that the only solution is the trivial one as long as $\tau = \pm v$ and for $\tau \neq \pm v$ no solution exists.

Case 2. If u and $\hat{\mathbf{c}}_2$ are given, respectively v and $\hat{\mathbf{c}}_1$ are unknown, we multiply with $\mathcal{R}(-v\hat{\mathbf{c}}_2) = \mathcal{R}^{-1}(v\hat{\mathbf{c}}_2)$ on the left and use (5) to obtain

$$u\hat{\mathbf{c}}_1 = \langle -v\hat{\mathbf{c}}_2, \mathbf{c} \rangle = \frac{\tau\hat{\mathbf{n}} - v\hat{\mathbf{c}}_2 + v\tau\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2}{1 + v\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)} \cdot \tag{12}$$

Certainly, in this case (8) is still valid as a condition for the existence of (6).

On the other hand, the scalar product of (12) with $\hat{\mathbf{c}}_2$ yields

$$u\cos\gamma = \frac{\tau\rho_2 - v}{1 + v\tau\rho_2} \cdot \tag{13}$$

We proceed similarly to the previous case by calculating the matrix element

$$\sigma_{22} = (\hat{\mathbf{c}}_2, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_2) = (\hat{\mathbf{c}}_2, \mathcal{R}(u\hat{\mathbf{c}}_1)\,\hat{\mathbf{c}}_2) = \cos\psi_1 + (1-\cos\psi_1)\cos^2\gamma \tag{14}$$

with $\psi_1 = 2\arctan(u)$ representing the Euler angle of the first rotation, which element is known. Substituting (14) into (13) leads to

$$v_\pm = \frac{\tau\rho_2 - u\cos\gamma_\pm}{1 + u\tau\rho_2\cos\gamma_\pm}, \qquad u\cos\gamma_\pm = \pm\frac{1}{\sqrt{2}}\sqrt{(u^2+1)\,\sigma_{22} + u^2 - 1} \tag{15}$$

together with the condition $\sigma_{22} \geq \cos\psi_1$.

Here, similarly to the previous case, for $\cos\gamma = 0$ ($\sigma_{22} = \cos\psi_1$), we have $v_+ = v_- = \tau\rho_2$.

3. Special cases. We consider separately the cases of half turns and quarter turns. For the former it is a necessity due to the problem of divergent scalar parameters, while for the latter, apart from its significance in various physical applications, our main motive is simplicity of expressions.

3.1. Half turns. If $\mathcal{R}(\mathbf{c})$ is a symmetric rotation of type (2), the expressions (7), (11), (12) and (15) are ill defined since the scalar parameter τ diverges. However, one may easily derive the corresponding expressions by considering the limit $\tau \to \infty$ and obtain for the first case

$$v\hat{\mathbf{c}}_2 = \lim_{\tau\to\infty} \langle \mathbf{c}, -u\hat{\mathbf{c}}_1 \rangle = \frac{\hat{\mathbf{n}} + u\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}}}{u(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)}, \tag{16}$$

where u should be substituted with the corresponding asymptotic solutions of (11)

$$u_\pm = \frac{1}{v\cos\gamma_\pm}\cdot$$

In this particular setting it is easy to see that the double root in the case $\sigma_{11} = \cos\psi_2$ corresponds to a half turn $\mathcal{O}(\hat{\mathbf{n}}) = \mathcal{R}(v\hat{\mathbf{c}}_2)\,\mathcal{O}(\hat{\mathbf{c}}_1)$ and (16) should be considered in the limit $u \to \infty$, namely

$$v\hat{\mathbf{c}}_2 = \lim_{u,\tau\to\infty} \langle \mathbf{c}, -u\hat{\mathbf{c}}_1 \rangle = \frac{\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}}}{(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)} \tag{17}$$

that is only possible if $v = \pm\tan\measuredangle(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1) = \pm(1+\rho_1^2)^{-1/2}$ while the choice of the sign depends on the orientation. In any other case, as long as $\cos\gamma \neq 0$, we end up with a pair of two equally valid solutions. In a similar way we obtain

$$u\hat{\mathbf{c}}_1 = \lim_{\tau\to\infty} \langle -v\hat{\mathbf{c}}_2, \mathbf{c} \rangle = \frac{\hat{\mathbf{n}} + v\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2}{v(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)}, \qquad v_\pm = \frac{1}{u\cos\gamma_\pm}\cdot \tag{18}$$

As for the case $v, \tau \to \infty$, we have

$$u\hat{\mathbf{c}}_1 = \lim_{v,\tau\to\infty} \langle -v\hat{\mathbf{c}}_2, \mathbf{c} \rangle = \frac{\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2}{(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)}, \qquad u = \pm\tan\measuredangle(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2) = \pm(1+\rho_2^2)^{-1/2}. \tag{19}$$

Another interesting case to investigate in this context is when τ is regular, but the rotation with unknown axis is a half turn – then we consider the corresponding limits to obtain $u_\pm = -(\tau\rho_1)^{-1}$ for the first case and $v_\pm = -(\tau\rho_2)^{-1}$ for the second one, which fully complies with the results obtained in [1].

Finally, the solutions in the cases $u, \tau \to \infty$ and $v, \tau \to \infty$ can be obtained from (7), respectively (12) as

$$(20)\quad v\hat{\mathbf{c}}_2 = \lim_{u\to\infty}\langle \mathbf{c}, -u\hat{\mathbf{c}}_1\rangle = \frac{\tau\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}} - \hat{\mathbf{c}}_1}{\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)}, \quad u\hat{\mathbf{c}}_1 = \lim_{v\to\infty}\langle -v\hat{\mathbf{c}}_2, \mathbf{c}\rangle = \frac{\tau\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2 - \hat{\mathbf{c}}_2}{\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)}.$$

The conditions of vanishing denominators in (11) and (15), that may also be retrieved from the above, we write in the form

$$(21)\quad \tau^2\rho_1^2\left((1+v^2)\,\sigma_{11} + v^2 - 1\right) = 2$$

and

$$(22)\quad \tau^2\rho_2^2\left((1+u^2)\,\sigma_{22} + u^2 - 1\right) = 2$$

respectively.

3.2. Degenerate solutions. If $\mathcal{R}(\mathbf{c})$ is a half turn itself, the condition $\cos\gamma = 0$ in the first case can be written as $\sigma_{11} = \cos\psi_2$, while in the second one we have $\sigma_{22} = \cos\psi_1$. We have another geometric condition $\rho_i = 0$, $i = 1, 2$ for Case 1 and Case 2 respectively, without that one only obtains the trivial solution. This situation does not entirely fit our construction – even if the above conditions are fulfilled, one still cannot determine the unknown axis of rotation due to the zero denominators in (20) in the limit $\tau \to \infty$. Hence, the procedure we have built here cannot be used for obtaining the value of the unknown scalar parameter directly from (11) or (15). Instead, considering the limit $\tau \to \infty$ and properly chosen scalar products which appear in (20), one obtains for the two cases

$$(23)\quad u = \frac{(\hat{\mathbf{c}}_1, \hat{\mathbf{n}} \times \hat{\mathbf{c}}_2)}{(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)}, \qquad v = \frac{(\hat{\mathbf{c}}_2, \hat{\mathbf{n}} \times \hat{\mathbf{c}}_1)}{(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)}.$$

Note that the vector $\hat{\mathbf{c}}_2$ in the former case (respectively $\hat{\mathbf{c}}_1$ in the latter) cannot be uniquely determined – it is only constrained to $\mathbb{S}^1$, since the two axes are perpendicular: $\cos\gamma = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2) = 0$ and their relative orientation with respect to $\hat{\mathbf{n}}$ determines the value of u (respectively v), which gives rise to a one-parameter set of solutions. Such degenerate solutions appear in the classical Euler decomposition setting as well and the phenomenon, considered a serious obstruction in engineering applications, is called *gimbal lock*. Although it is usually defined in a slightly different context, the situation is quite similar to what we encounter here and both phenomena can be explained with a singularity of a projection map – the rank drops at the singular points, causing loss of a degree of freedom.

For the sake of brevity, however, we omit detailed discussions about the topological aspects of the mappings, constructed in this paper, and concentrate on explicit formulae instead.

3.3. Quarter turns. Here we apply the expressions derived in the previous section to the particular case of quarter turns. More precisely, we are looking for a decomposition of the type

$$(24)\quad \mathcal{R}(\mathbf{c}) = \mathcal{Q}(\hat{\mathbf{c}}_2)\mathcal{R}(u\hat{\mathbf{c}}_1)$$

or, alternatively

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(v\hat{\mathbf{c}}_2)\mathcal{Q}(\hat{\mathbf{c}}_1). \tag{25}$$

Note that each of the matrices $\mathcal{R}(\mathbf{c})$, $\mathcal{R}(u\hat{\mathbf{c}}_1)$ and $\mathcal{R}(v\hat{\mathbf{c}}_2)$ can be a half turn itself. Such solutions are considered at the end of this section. At present, we focus our attention to the "generic" case $\varphi, \psi_{1,2} \neq \pi$.

The unknowns, or the "free parameters" in the decomposition are the axis of the quarter turn and the scalar parameter of the remaining rotation for which the axis is given. In other words, we map an element $\mathbf{c} \in \mathbb{RP}^3$ with a pair of parameters in $\mathbb{S}^2 \times \mathbb{S}^1$, slightly different from the standard ones: $(\hat{\mathbf{c}}_2, u)$ and $(\hat{\mathbf{c}}_1, v)$ respectively for the two cases.

Let us begin with $\mathcal{R}(\mathbf{c}) = \mathcal{Q}(\hat{\mathbf{c}}_2)\,\mathcal{R}(u\hat{\mathbf{c}}_1)$. Basically, we simply need to apply (7) and (11) with $v = 1$ in order to obtain

$$\hat{\mathbf{c}}_2 = \langle \mathbf{c}, -u\hat{\mathbf{c}}_1 \rangle = \frac{\tau\hat{\mathbf{n}} - u\hat{\mathbf{c}}_1 + u\tau\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}}}{1 + u\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)} \tag{26}$$

and

$$u_\pm = \frac{\tau\rho_1 \mp \sqrt{\sigma_{11}}}{1 \pm \tau\rho_1\sqrt{\sigma_{11}}}\cdot \tag{27}$$

Note that the condition that guarantees real solutions is reduced in this case to $\sigma_{11} \geq 0$. The two solutions, substituted in (26), determine two unit vectors $\hat{\mathbf{c}}_2^\pm$, or two possible axes for the quarter turn solution.

Obviously, in the case $\mathcal{R}(\mathbf{c}) = \mathcal{Q}(\hat{\mathbf{n}})$ one of these solutions is trivial: $u = 0$, $\hat{\mathbf{c}}_2 = \hat{\mathbf{n}}$ and if $\sigma_{11} = 0$, we end up with a double root $u_+ = u_- = \rho_1$.

Similarly, considering (12) and (15) for $u = 1$ one gets

$$\hat{\mathbf{c}}_1 = \langle -v\hat{\mathbf{c}}_2, \mathbf{c} \rangle = \frac{\tau\hat{\mathbf{n}} - v\hat{\mathbf{c}}_2 + v\tau\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2}{1 + v\tau(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)} \tag{28}$$

and

$$v_\pm = \frac{\tau\rho_2 \mp \sqrt{\sigma_{22}}}{1 \pm \tau\rho_2\sqrt{\sigma_{22}}}\cdot \tag{29}$$

As for the asymptotic behaviour (the cases involving a half turn) from (16) and (18) considered for $v = 1$, respectively $u = 1$, we easily obtain

$$\hat{\mathbf{c}}_2 = \lim_{\tau\to\infty} \langle \mathbf{c}, -u\hat{\mathbf{c}}_1 \rangle = \frac{\hat{\mathbf{n}} + u\hat{\mathbf{c}}_1 \times \hat{\mathbf{n}}}{u(\hat{\mathbf{n}}, \hat{\mathbf{c}}_1)} \tag{30}$$

with the obvious solution for the scalar parameter

$$u_\pm = \pm\frac{1}{\sqrt{\sigma_{11}}}$$

in the former case and similarly

$$\hat{\mathbf{c}}_1 = \lim_{\tau\to\infty} \langle -v\hat{\mathbf{c}}_2, \mathbf{c} \rangle = \frac{\hat{\mathbf{n}} + v\hat{\mathbf{n}} \times \hat{\mathbf{c}}_2}{v(\hat{\mathbf{n}}, \hat{\mathbf{c}}_2)} \tag{31}$$

leading to

$$v_\pm = \pm\frac{1}{\sqrt{\sigma_{22}}}$$

in the latter one.

The conditions under which the unknown parameter tends to infinity can be written here as $\tau^2\rho_i^2\sigma_{kk} = 1$, $k = 1, 2$ for the two cases under consideration.

We also note that degenerate solutions are not possible in this case, while one may encounter double roots when the corresponding diagonal element of σ vanishes. Moreover, these double roots correspond to half turns in the case $\tau \to \infty$, as it can be easily seen from the above.

4. Quaternion parametrization. We can make use of the well known local isomorphism between $\mathsf{SO}(3)$ and $\mathsf{SU}(2)$ to obtain another interpretation of the above relations. As it is usually being done (see [5,6,11]), we represent $\mathsf{SU}(2) \cong \mathbb{S}^3$ as the group of quaternions with unit norm $\mathsf{Sp}(1)$

$$\mathbb{S}^3 = \{\zeta = z + wj\,;\ \ z, w \in \mathbb{C},\ j^2 = -1,\ |\zeta|^2 = 1\}$$

and let it act in its Lie algebra of skew-hermitian matrices via the adjoint representation Ad_ζ, which can be viewed as a norm-preserving automorphism of $\mathbb{R}^3$. One may use the notation $\boldsymbol{\zeta} \in \mathbb{R}^3$ for the *imaginary*, or *vector* part of the quaternion ζ, i.e., the vector with *Cartesian* coordinates ζ_i, $i = 1, 2, 3$. Then we write $\zeta = (\zeta_0, \boldsymbol{\zeta})$ and refer to ζ_0 as the *real* or *scalar* part of ζ. With these notations the corresponding rotation matrix can be written as

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\,\mathcal{I} + 2\,\boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^t + 2\,\zeta_0\,\boldsymbol{\zeta}^\times. \tag{32}$$

Comparing the latter with (1), it is easy to conclude that if ζ acts as a rotation by an angle φ about the unit vector $\hat{\mathbf{n}}$, then

$$\zeta_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\,\hat{\mathbf{n}} \tag{33}$$

and therefore $\mathcal{R}(\zeta)$ is a half turn for a purely imaginary quaternion ($\zeta_0 \equiv 0$) and the identity transformation in the scalar case ($\zeta_0 \equiv 1$).

The correspondence between vector and quaternion parameters is given by the stereographic projection of $\mathbb{S}^3$ onto $\mathbb{RP}^3$

$$\zeta_0^2 = 1 - \boldsymbol{\zeta}^2 = \frac{1}{1 + \mathbf{c}^2}, \qquad \boldsymbol{\zeta} = \zeta_0\,\mathbf{c}. \tag{34}$$

Note that whenever ζ is a solution, so is $-\zeta$, since the quaternion correspondence built here comes from a two-fold cover. In order to keep notations simple, we choose to write only one of these solutions – usually the one with a positive real part. The composition law of quaternion parameters is given by

$$\zeta = \langle \eta, \xi \rangle = \eta\,\xi \tag{35}$$

in which we may use the multiplication table of $\mathfrak{su}(2)$ (see [4,5]) and separate the scalar and vector parts

$$\zeta = \eta\,\xi = (\zeta_0,\,\boldsymbol{\zeta}) = (\eta_0\xi_0 - (\boldsymbol{\eta}, \boldsymbol{\xi}),\ \eta_0\,\boldsymbol{\xi} + \xi_0\,\boldsymbol{\eta} + \boldsymbol{\eta} \times \boldsymbol{\xi})\,. \tag{36}$$

Notice also that one can easily derive (5) using stereographic projection. We refer to [2] for generalizations and applications of quaternions to inertial navigation problems.

5. Factorizations. Let $\xi = (\xi_0, \boldsymbol{\xi})$ be the quaternion parameter, corresponding to $\mathcal{R}(u\hat{\mathbf{c}}_1)$ and $\eta = (\eta_0, \boldsymbol{\eta})$ – the one corresponding to $\mathcal{R}(v\hat{\mathbf{c}}_2)$. If v and $\hat{\mathbf{c}}_1$ are known and u, $\hat{\mathbf{c}}_2$ – to be determined, this means that we are given $\hat{\boldsymbol{\xi}}$ – the unit vector, associated with $\boldsymbol{\xi}$ and $v = \dfrac{|\boldsymbol{\eta}|}{\eta_0}$, but need to determine $\hat{\boldsymbol{\eta}}$, ξ_0 and $|\boldsymbol{\xi}| = \sqrt{1-\xi_0^2}$ respectively.

For the former, as long as the generic case is concerned, we use the solutions of (11) for the derivation of

$$(37)\quad \xi_0^{\pm} = \frac{1 + v\tau\rho_1\cos\gamma_{\pm}}{\sqrt{(1+v^2\cos^2\gamma_{\pm})(1+\tau^2\rho_1^2)}}, \quad \boldsymbol{\xi}^{\pm} = \frac{\tau\rho_1 - v\cos\gamma_{\pm}}{\sqrt{(1+v^2\cos^2\gamma_{\pm})(1+\tau^2\rho_1^2)}}\,\hat{\mathbf{c}}_1,$$

where $v\cos\gamma_{\pm}$ is calculated according to (11). As for the latter, the unit vector $\hat{\boldsymbol{\eta}} = \hat{\mathbf{c}}_2$ is determined from (7) and then the quaternion η is fully restored by

$$(38)\qquad \eta_0 = (1+v^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\eta}^{\pm} = \eta_0 v\,\hat{\mathbf{c}}_2^{\pm} = \frac{v}{\sqrt{1+v^2}}\,\hat{\mathbf{c}}_2^{\pm}.$$

In a similar way we find the corresponding solutions for the case when $u = \dfrac{|\boldsymbol{\xi}|}{\xi_0}$ and $\hat{\mathbf{c}}_2 = \hat{\boldsymbol{\eta}}$ are given, and then use (15) to obtain

$$(39)\quad \eta_0^{\pm} = \frac{1 + u\tau\rho_2\cos\gamma_{\pm}}{\sqrt{(1+u^2\cos^2\gamma_{\pm})(1+\tau^2\rho_2^2)}}, \quad \boldsymbol{\eta}^{\pm} = \frac{\tau\rho_2 - u\cos\gamma_{\pm}}{\sqrt{(1+u^2\cos^2\gamma_{\pm})(1+\tau^2\rho_2^2)}}\,\hat{\mathbf{c}}_2$$

with $u\cos\gamma_{\pm}$ determined by (15), and the solutions of (12) for

$$(40)\qquad \xi_0 = (1+u^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\xi}^{\pm} = \xi_0 u\,\hat{\mathbf{c}}_1^{\pm} = \frac{u}{\sqrt{1+u^2}}\,\hat{\mathbf{c}}_1^{\pm}.$$

We may also obtain the quaternion representation for the cases described in Subsection 3.1 and Subsection 3.3 by substituting the corresponding solutions in the above expressions for ξ and η. The cases, involving a half turn, may easily be resolved by studying the asymptotic behaviour of the generic solutions.

If, on the other hand, η corresponds to a quarter turn, we have

$$\eta_0 = |\boldsymbol{\eta}| = \frac{1}{\sqrt{2}}, \qquad \boldsymbol{\eta}^{\pm} = \frac{1}{\sqrt{2}}\,\hat{\mathbf{c}}_2^{\pm}.$$

The full solutions in this case are given by

$$\xi_0^{\pm} = \frac{1 \pm \tau\rho_1\sqrt{\sigma_{11}}}{\sqrt{(1+\sigma_{11})(1+\tau^2\rho_1^2)}}, \qquad \boldsymbol{\xi}^{\pm} = \frac{\tau\rho_1 \mp \sqrt{\sigma_{11}}}{\sqrt{(1+\sigma_{11})(1+\tau^2\rho_1^2)}}\,\hat{\mathbf{c}}_1$$

for the decomposition (24) and

$$\eta_0^{\pm} = \frac{1 \pm \tau\rho_2\sqrt{\sigma_{22}}}{\sqrt{(1+\sigma_{22})(1+\tau^2\rho_2^2)}}, \qquad \boldsymbol{\eta}^{\pm} = \frac{\tau\rho_2 \mp \sqrt{\sigma_{22}}}{\sqrt{(1+\sigma_{22})(1+\tau^2\rho_2^2)}}\,\hat{\mathbf{c}}_2$$

respectively

$$\xi_0 = \frac{1}{\sqrt{2}}, \qquad \boldsymbol{\xi}^{\pm} = \frac{1}{\sqrt{2}}\,\hat{\mathbf{c}}_1^{\pm}$$

for the decomposition (25).

The above expressions are reduced in the case $\tau \to \infty$ to

$$\xi_0^{\pm} = \pm\sqrt{\frac{\sigma_{11}}{1+\sigma_{11}}}, \qquad \boldsymbol{\xi}^{\pm} = \frac{\operatorname{sgn}(\rho_1)}{\sqrt{1+\sigma_{11}}}\,\hat{\mathbf{c}}_1$$

and

$$\eta_0^{\pm} = \pm\sqrt{\frac{\sigma_{22}}{1+\sigma_{22}}}, \qquad \boldsymbol{\eta}^{\pm} = \frac{\operatorname{sgn}(\rho_2)}{\sqrt{1+\sigma_{22}}}\,\hat{\mathbf{c}}_2.$$

Note that we can omit the factors $\operatorname{sgn}(\rho_i)$ since the quaternion corresponding to a given rotation is determined up to a sign anyway – for the same reason we may choose the scalar part to be positive and leave the factor "±" for the vector one.

The expressions for the generic case can be obtained from the above by replacing σ_{11}, σ_{22} with $v^2\cos^2\gamma_{\pm}$ and $u^2\cos^2\gamma_{\pm}$ respectively.

In order to have the full quaternion correspondence, however, we need to express the solutions $u_{\pm}$ $\hat{\mathbf{c}}_2^{\pm}$ and $v_{\pm}$, $\hat{\mathbf{c}}_1^{\pm}$ in terms of ζ. Using (32) and (34), we easily obtain

$$u_{\pm} = \frac{(\boldsymbol{\zeta}, \hat{\mathbf{c}}_1) - \zeta_0 v\cos\gamma_{\pm}}{\zeta_0 + (\boldsymbol{\zeta}, \hat{\mathbf{c}}_1)\, v\cos\gamma_{\pm}}, \qquad v\hat{\mathbf{c}}_2^{\pm} = \frac{\boldsymbol{\zeta} - \zeta_0 u_{\pm}\hat{\mathbf{c}}_1 + u_{\pm}\hat{\mathbf{c}}_1 \times \boldsymbol{\zeta}}{\zeta_0 + u_{\pm}(\boldsymbol{\zeta}, \hat{\mathbf{c}}_1)} \tag{41}$$

and

$$v_{\pm} = \frac{(\boldsymbol{\zeta}, \hat{\mathbf{c}}_2) - \zeta_0 u\cos\gamma_{\pm}}{\zeta_0 + (\boldsymbol{\zeta}, \hat{\mathbf{c}}_2)\, u\cos\gamma_{\pm}}, \qquad u\hat{\mathbf{c}}_1^{\pm} = \frac{\boldsymbol{\zeta} - \zeta_0 v_{\pm}\hat{\mathbf{c}}_2 + v_{\pm}\boldsymbol{\zeta} \times \hat{\mathbf{c}}_2}{\zeta_0 + v_{\pm}(\boldsymbol{\zeta}, \hat{\mathbf{c}}_2)}, \tag{42}$$

where $\mathbf{c}_1 = u\hat{\mathbf{c}}_1$, $\mathbf{c}_2 = v\hat{\mathbf{c}}_2$, while $v\cos\gamma_{\pm}$, $u\cos\gamma_{\pm}$ are calculated according to (11) and (15) respectively, and the matrix entry there is given by

$$\sigma_{kk} = \zeta_0^2 - \boldsymbol{\zeta}^2 + 2(\boldsymbol{\zeta}, \hat{\mathbf{c}}_k)^2, \qquad k = 1, 2.$$

In particular, for a quarter turn we have $|v\cos\gamma| = \sqrt{\sigma_{11}}$ in the first case, and $|u\cos\gamma| = \sqrt{\sigma_{22}}$ in the second one.

On the other hand, with the help of (36), we may easily write the corresponding expressions for ξ and η in terms of ζ. This can be viewed as a double valued map $\mathbb{S}^3 \to \mathbb{S}^2 \times \mathbb{S}^1$, $\zeta \to (\hat{\boldsymbol{\eta}}^{\pm}, \xi_0^{\pm})$, respectively $\zeta \to (\hat{\boldsymbol{\xi}}^{\pm}, \eta_0^{\pm})$. The explicit formulas are

$$\begin{aligned} \xi_0^{\pm} &= \frac{\zeta_0\eta_0 \pm (\boldsymbol{\zeta}, \hat{\boldsymbol{\xi}})\sqrt{\zeta_0^2 + (\boldsymbol{\zeta}, \hat{\boldsymbol{\xi}})^2 - \eta_0^2}}{\zeta_0^2 + (\boldsymbol{\zeta}, \hat{\boldsymbol{\xi}})^2}, \\ \boldsymbol{\eta}^{\pm} &= \xi_0^{\pm}\boldsymbol{\zeta} - \zeta_0\boldsymbol{\xi}^{\pm} + \boldsymbol{\xi}^{\pm} \times \boldsymbol{\zeta}, \qquad \boldsymbol{\xi}^{\pm} = \sqrt{1 - \xi_0^{\pm 2}}\,\hat{\boldsymbol{\xi}} \end{aligned} \tag{43}$$

for the decomposition (24) and for (25) we obtain in a similar way

$$\begin{aligned} \eta_0^{\pm} &= \frac{\zeta_0\xi_0 \pm (\boldsymbol{\zeta}, \hat{\boldsymbol{\eta}})\sqrt{\zeta_0^2 + (\boldsymbol{\zeta}, \hat{\boldsymbol{\eta}})^2 - \xi_0^2}}{\zeta_0^2 + (\boldsymbol{\zeta}, \hat{\boldsymbol{\eta}})^2}, \\ \boldsymbol{\xi}^{\pm} &= \eta_0^{\pm}\boldsymbol{\zeta} - \zeta_0\boldsymbol{\eta}^{\pm} + \boldsymbol{\zeta} \times \boldsymbol{\eta}^{\pm}, \qquad \boldsymbol{\eta}^{\pm} = \sqrt{1 - \eta_0^{\pm 2}}\,\hat{\boldsymbol{\eta}}. \end{aligned} \tag{44}$$

Concerning the solutions, corresponding to quarter turns, we only substitute $\eta_0 = \dfrac{1}{\sqrt{2}}$ or $\xi_0 = \dfrac{1}{\sqrt{2}}$, while the rest of the expressions remain the same.

Note that the presence of half turns in this representation does not constitute any obstruction – it merely makes a certain coordinate to vanish. Nevertheless, the degenerate solutions, described in Section 3, may still appear: setting $\zeta_0 = \eta_0 = 0$ and $(\boldsymbol{\zeta}, \hat{\boldsymbol{\xi}}) = 0$ in (43) lead to indeterminacy for ξ_0, so $\hat{\boldsymbol{\eta}}$ is an arbitrary unit vector in the plane normal to $\boldsymbol{\xi}$ and $\xi_0 = (\boldsymbol{\zeta}, \hat{\boldsymbol{\eta}}) = \rho_2$ is its projection along $\boldsymbol{\zeta}$. Similarly, whenever $\zeta_0 = \xi_0 = 0$ and $(\boldsymbol{\zeta}, \hat{\boldsymbol{\eta}}) = 0$ in (44), the unknown scalar part of the quaternion in the decomposition is determined through the projection of an arbitrary (within the plane, perpendicular to $\boldsymbol{\eta}$) unit vector $\hat{\boldsymbol{\xi}}$, namely $\eta_0 = -(\boldsymbol{\zeta}, \hat{\boldsymbol{\xi}}) = -\rho_1$.

6. Final remarks. There are many possible extensions and generalizations of the main construction explored in this article. For instance, one may investigate the applications to hyperbolic geometry given by the standard Lie group homomorphisms and isomorphisms [4]. Another route is to consider decompositions of the three-dimensional rotations or Lorentz transformations into three factors, in which the unknowns are not necessarily the angles or scalar parameters as in the classical Euler setting, but using more complicated combinations of geometrical data for both the axes and the angles.

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