About Parametric Representations of SO(n) Matrices and Plane Rotations

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Abstract. Since a long time the group SO(n) is of a great interest in physics (space relativity theory, quantum electrodynamics, theory of elementary particles) and mechanics because of its numerous applications to problems of monitoring of unknown nonlinear systems. The present paper treats the basic theory of this group and it is shown that any transformation of the group SO(n) may be presented as a product of plane transformations in clear analytical forms, appropriate for practical applications. The approach presented here is inspired by the close analogy of plane rotations with the vector-parameterization of the SO(3) group.

Keywords: SO(n) group, SO(3) group, Lie groups, Lie algebras, parameterizations of the rotation group, plane rotations, vector-parameter
PACS: 02.20.Qs, 02.20.Tw, 06.30.Bp, 06.30.Gv, 45.20.dc, 45.40.Bb

INTRODUCTION

The group SO(n) is a generalization of the SO(3) rotation group acting in $\mathbb{R}^n$. Since $\mathbb{R}^3$ space is the real space where we live, and where all laws of classical mechanics are valid, the experience with the investigations of the motions in it is helpful to study the motions in higher dimensions. Here is the place to stress the special attention on the group SO(3) since it is a very important in modelling and control of a mechanical system in $\mathbb{R}^3$ and its kinematical description [2]. It is well known that the rigid-body motion in $\mathbb{R}^3$ is described by the Euclidean group E(3), and that the SO(3) group cannot be avoided in the representation of orientations. The appropriate parameterization of SO(3) is one of the most important practical problem in mechanics because it has a great influence over the overall efficiency of all methods [3], [16]. Angular velocity or momentum information is required by the most control strategies. It could be obtained using the derivatives of various orientation parameters like Euler and Bryant angles, Euler or Cayley-Klein parameters, quaternions [2], [5], *etc.*, or the so called vector-parameter (called also Rodrigues or Gibbs vector), which as an element of a Lie group, has a very nice and clear properties and simplifies the treatment of many problems [9], [11], [12]. It is worth to be mentioned that there is an analogy between the rigid body description through vector-parameter and this one realized on the base of screw operators. The intrinsic mathematical formalism in physical rigid body motions description is presented with the use of affine geometry together with Lie group theory and it is used for description of the kinematic pairs.

After introducing the vector-parameter in connection with representation theory of the Lorentz group in the special relativity theory [4], different group parameterizations of the rotational motion for higher dimensions and their after-effects are investigated. For example, an useful algorithm for numerical parametrical presentation of SO(n) group may be found in [17], which is applied in [19] for the purposes of quadratic stability analysis of cognitive - oriented control. There is also a great interest in using SO(n) for $n \geq 4$ in the theory of elementary particles. In the general case (when n is great) the expressions for the orthogonal matrices are very complicated, but as it is shown latter in the paper, in every group SO(n) may be found a subgroup of transformations which may be parameterized in a simple and universal way which does not depend on the dimension n of the vector space.

Plane rotations appear in many classical and quantum mechanical analysis which lead to considerations of the spectrum and eigenvectors of either 3×3 or 4×4 real symmetric matrices. The story starts with the Jacobi's method for solving the eigenvalue problem for a concrete 8×8 symmetric matrix that arises in his studies on dynamics. Jacobi diagonalizes the above matrix by performing a sequence of orthogonal similarity transformations and his method is relevant and effective in all dimensions (see [7] for numerical counterpart). Each transformation is a plane rotation, chosen so that the induced similarity diagonalizes some 2×2 principal submatrix, moving the weight of the annihilated elements onto the diagonal. Performing the same procedure in lower dimensions has a lot of specificity. *E.g.*, using the isomorphism between 4×4 orthogonal matrices and algebra of quaternions [1] present a construction of an orthogonal

Application of Mathematics in Technical and Natural Sciences
AIP Conf. Proc. 1487, 280-287 (2012); doi: 10.1063/1.4758969

similarity that acts directly on 2×2 blocks and diagonalizes a 4×4 symmetric matrix. This problem appears in many concrete situations in physics, mechanics, crystallography, elasticity, hydromechanics, robotics, etc., where one has to deal with various symmetric matrices.
The paper is organized as follows: After a short introduction clarifying the aim of the paper and in what kind of problems SO(n) group is used, the next section deals with SO(n) general theory. In particular, any element of the group SO(n) is presented as a product of not more than $[n/2]$ plane transformations and it is noted that the theory of plane rotations reviewed in the text is analogous to the vector-parameterization of the rotation group SO(3), that is treated in a separate section. The analytical form of $n\times n$ orthogonal matrices is presented in the final section.

SO(N) IN THE HAMILTON - CAYLEY FORM

A **Lie group** is a set G such that: 1) G is a group; 2) G is a smooth manifold; 3) the group operations of composition and inversion are smooth maps of G into itself relative to the manifold structure defined in 2).
SO(n) is a Lie group. The matrices in SO(n) present the rotational motions and as a set are defined as follows

$$\mathrm{SO}(n) = \{O \in \mathrm{Mat}(n,\mathbb{R});\ \det O = 1,\ OO^T = I\} \tag{1}$$

where $\mathrm{Mat}(n,\mathbb{R})$ is the group of $n\times n$ matrices with elements in $\mathbb{R}$ together with its Lie algebra (i.e., its infinitesimal generators) consisting of the skew-symmetric $n\times n$ matrices. If A belongs to the Lie algebra of SO(n), the matrix $I-A$ is invertible (see [10], [11]). The Hamilton-Cayley formula provides in general the connection between the Lie algebra and the group, and therefore every orthogonal $n\times n$ matrix $O\in$ SO(n) $(OO^T = I, \det O = I)$ in n - dimensional vector space (real or complex) can be written in the form [5], [18], [15]

$$O = O(A) = (I+A)(I-A)^{-1} = (2I-(I-A))(I-A)^{-1} = 2(I-A)^{-1} - I. \tag{2}$$

It follows that it can be easily inverted and in this way one obtains $A = (O-I)(O+I)^{-1}$, $\quad A^T = (O^T-I)(O^T+I)^{-1}$, or

$$A = \frac{1}{2}(A-A^T) = (O-O^T)(2I+O+O^T)^{-1}. \tag{3}$$

The last is fulfilled provided that $\det(I+O)\neq 0$, i.e., $|I+O|\neq 0$, which is satisfied since one has $O+I = 2(I-A)^{-1}$ and $|O+I| = 2^n|I-A|^{-1}$. Hence, $|O+I| = 0$ only when the elements of the matrix A are very large. For the matrix $(I-A)^{-1}$ we may write the Neumann series, namely: $(I-A)^{-1} = I+A+A^2+A^3+\ldots$. According to the theorem of Hamilton - Cayley every matrix is a root of its characteristic polynomial, which degree is equal to the order n of the matrix. Hence, the n-th and higher degrees of the matrix O are expressed through the lower ones

$$O = a_1A^{n-1} + a_2A^{n-2} + \ldots a_{n-1}A + a_n. \tag{4}$$

The independent parameters of the matrix $A = A_{ij}$ are $n(n-1)/2$ and they are parameters which define the special orthogonal transformation SO(n). Generally, the matrix O may be presented as $O = f(A)/f(-A)$, where $f(A)$ is any bounded function for which $f(A)\neq f(-A)$ and $|f(A)|\neq 0$. For example, such function is $f(A) = \exp A/2$ and therefore $O = \exp A$ (for more details see e.g. [14]).
From the general theory we have $A\psi = \lambda\psi$, where ψ and λ are respectively the eigenvector and the eigenvalue. Then we may present the determinant of the matrix $\lambda I - A$ in the well-known polynomial form

$$|\lambda - A| = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} - \ldots + (-1)^n a_n = 0 \tag{5}$$

where the following equations are valid

$$a_1 = \lambda_1+\lambda_2+\ldots+\lambda_n = A_{11}+A_{22}+\ldots+A_{nn} = A_t \tag{6}$$

$$a_n = \lambda_1\lambda_2\ldots\lambda_n \tag{7}$$

$$a_2 = \lambda_1\lambda_2+\lambda_1\lambda_3+\ldots+\lambda_{n-1}\lambda_n = \frac{1}{2}[(\lambda_1+\lambda_2+\ldots+\lambda_n)^2 - (\lambda_1^2+\lambda_2^2+\ldots+\lambda_n^2)] \tag{8}$$

$$A_t^k = (A^k)_t = \lambda_1^k+\lambda_2^k+\ldots+\lambda_n^k \tag{9}$$

$$a_2 = \frac{1}{2}[(A_t)^2 - (A^2)_t], \qquad A_t \text{ means the trace of the matrix } A. \tag{10}$$

For a matrix A of second order one has: $A^2+A_tA+|A|=0$. Analogically, identities for higher order matrices may be obtained, and having in mind the Hamilton - Cayley equation for the skew-symmetric matrices, they can be simplified significantly, and in the special cases of $n=2,3,4,5,6$ look like

$$
\begin{aligned}
A^2+|A| &= 0 && (11)\\
A^3-\frac{1}{2}(A^2)_tA &= 0 && (12)\\
A^4-\frac{1}{2}(A^2)_tA^2+|A| &= 0 && (13)\\
A^5-\frac{1}{2}(A^2)_tA^3+\frac{1}{2}[((A^2)_t)^2-(A^4)_t]A &= 0 && (14)\\
A^6-\frac{1}{2}(A^2)_tA^4+\frac{1}{2}[((A^2)_t)^2-(A^4)_t]A^2-|A| &= 0. && (15)
\end{aligned}
$$

PLANE ORTHOGONAL TRANSFORMATIONS

The matrix $O=O(A)$ in (2) may be presented in the following way

$$
\begin{aligned}
O &= O(A)=2\sum_{r=0}^{k-1}b_{2r}(I-A)A^{2(k-1-r)}-I, \qquad n=2k\\
O &= O(A)=I+2\sum_{r=0}^{k-1}b_{2r}(I-A)A^{2(k-r)+1}, \qquad n=2k+1
\end{aligned}
\tag{16}
$$

where

$$
b_{2r}=(\sum_{m=0}^{r}a_{2m})/(\sum_{m=0}^{k}a_{2m}). \tag{17}
$$

Here

$$
\begin{aligned}
a_{2m} &= (-1)^mS^m=(-1)^m\sum_{i_r=1}^{k}\lambda_{i_1}^2\lambda_{i_2}^2\ldots\lambda_{i_r}^2\ldots\lambda_{i_m}^2\\
&= -[a_{2(m-1)}s_1+a_{2(m-2)}s_2+\ldots+a_2s_{m-1}+s_m], \qquad i_1\neq i_2\neq\ldots\neq i_r\ldots\neq i_m
\end{aligned}
\tag{18}
$$

are the coefficients of the characteristic (minimal) equation of a skew-symmetric $n\times n$ matrix of a general type, expressed by elementary symmetrical polynomials s_m via the squares of the eigenvalues $\lambda_{2i}=-\lambda_{2i+1},\ i=1,2,\ldots k$

$$
s_m=\sum_{i=1}^{k}\lambda_i^{2m}=\frac{1}{2}(A^{2m})_t, \qquad m=0,1,\ldots,k. \tag{19}
$$

It is easy to be seen that the equations (16 - 19) are alternative general form of those given in the previous section. The skew-symmetric matrix A may be written in the form of linear combinations

$$
A=\sum_{i=1}^{m}A_{0i}=\sum_{i=1}^{m}\alpha_i(z'^i.z^i-z^i.z'^i), \qquad m\leq N=\frac{n(n-1)}{2} \tag{20}
$$

of the plane matrices

$$
A_0=\alpha(z'.z-z.z'), \qquad A_{0[kl]}=-A_{0[lk]}=\alpha(z'_k.z_l-z_k.z'_l), \qquad k,l=1,2,\ldots,n \tag{21}
$$

where $z.y=(z_ky_l)$ means diadic matrix composed of n-dimensional (real or complex when SO(n, $\mathbb{C}$) is considered) vectors z and y – the so called divisors of the matrix A_0, and α is a normalizing coefficient.

Using some tensor algebra we may introduce the matrix of rang $n-2$, the so called dual skew-symmetric matrix $A^{\times}$ of A as follows

$$A^{\times}_{i_1 i_2 \ldots i_{n-2}} = \varepsilon_{i_1 i_2 \ldots i_{n-2} i_{n-1} i_n} A_{i_{n-1} i_n}, \qquad i_k = 1, 2, \ldots n \tag{22}$$

where $\varepsilon_{i_1 i_2 \ldots i_{n-2} i_{n-1} i_n}$ are the Levi - Civita symbols in n - dimensional space by which one defines also the product $AB^{\times}$

$$(AB^{\times})_{i_1 i_2 \ldots i_{n-3} j} = \varepsilon_{i_1 \ldots i_n} A_{i_{n-2} i_{n-1}} B_{i_n j}. \tag{23}$$

Then the necessary and sufficient condition the matrix A to be a simple one, *i.e.,* $A = -A^T$ is equivalent the matrix products given below to be equal to the zero matrix, namely

$$AA^{\times} = A^{\times} A = 0. \tag{24}$$

In a similar way one can define the product $A^{\times} B^{\times}$ and to obtain the relations

$$AB + B^{\times} A^{\times} = \frac{1}{2}(AB)_t, \qquad A^2 + (A^{\times})^2 = \frac{1}{2}(A^2)_t. \tag{25}$$

From the second equation of (25) follows the minimal equation

$$A_0^3 = \frac{1}{2}(A_0^2)_t A_0, \qquad \frac{1}{2}(A_0^2)_t = s_1 = \lambda^2. \tag{26}$$

The inverse statement is also valid: every $n \times n$ matrix $A = -A^T$, which obeys to (26) and for which $s_1 \neq 0$ is simple. In this manner we may conclude that the conditions (26) and $s_1 \neq 0$ are necessary and sufficient for the matrix A to be a simple one.
The orthogonal transformation $O_0 = O(A_0)$ (see the equations 16), which is defined through the simple matrix A_0, which fulfills (26) may be written in the general universal form

$$O(A_0) = O_0 = I + \frac{2}{1 - s_1}(A_0 + A_0^2), \qquad 1 - s_1 \neq 0, \qquad s_1 = \frac{1}{2}(A_0^2)_t \tag{27}$$

which coincides with the form of any transformation of the group SO(3). For the matrix $O_0 = O(A_0)$ from (27) is valid the minimal equation

$$O_0^3 = (\gamma - 1)(O_0^2 - O_0) + I, \qquad \gamma = 4 - n + (O_0)_t \tag{28}$$

and conversely, the matrix $A = -A^T$, defining the transformation $O = O(A)$ which obeys to (26) is simple. In this case the general relation (3) becomes

$$A = A_0 = (O_0 - O_0^T)(2I + O_0 + O_0^T)^{-1} = (O_0 - O_0^T)/(4 - n + (O_0)_t). \tag{29}$$

Definition: The orthogonal transformation $O = O(A_0)$ defined in (27) and using the skew - symmetric matrix - parameters A_0 is **plane** rotation. This name is natural, because the vectors z, z', through which according to (21) the matrix A_0 is expressed, define some plane (flat) in n - dimensional space. The product of two plane orthogonal transformations is also a plane one

$$O_0' O_0 = O(A_0') O(A_0) = O(A_0'') = O_0''. \tag{30}$$

In this case the matrices A_0, A_0' have to satisfy the condition

$$A_0 A_0'^{\times} + A_0' A_0^{\times} = 0 \tag{31}$$

and since

$$(AB^{\times} + BA^{\times})_{i_1 i_2 \ldots i_{n-3} j} = \frac{1}{2}\delta_{i_1 j}(AB^{\times})_{m i_2 \ldots i_{n-3} m}, \qquad \delta_{i_1 j} - \text{Christoffel symbol} \tag{32}$$

we may write

$$(A_0 A_0'^{\times})_{ji_2 \dots i_{n-3} j} = (A_0'^{\times} A_0)_{ji_2 \dots i_{n-3} j} = 0. \tag{33}$$

Having in mind the relation (29), for the resultant matrix - parameter $A'' = A_0''$ the following expression is obtained

$$A_0'' = < A_0', A_0 > = \frac{A_0' + A_0 + [A_0', A_0]}{1 + \frac{1}{2}(A_0' A_0)_t} \tag{34}$$

which is analogical to the composition law of the vector - parameters of the SO(3) group.
If we set for α in (21) to be equal to $\alpha = (z^2 + zz')^{-1}$, at the conditions $z^2 = z'^2 \neq 0, \quad 1 - s_1 \neq 0$, we get

$$A_0 = \frac{z'.z - z.z'}{z^2 + zz'}, \qquad s_1 = \frac{zz' - z^2}{zz' + z^2}. \tag{35}$$

If we substitute (35) in (27), we obtain the matrix of so called plane orthogonal transformation

$$O_0 = O(A_0) = I - 2e.e + 2e_0'.e_0' = O[z', z] \tag{36}$$

which realizes the transition

$$O[z', z]\, z = z'. \tag{37}$$

Here

$$e_0 = \frac{z}{\sqrt{z^2}}, \qquad e_0' = \frac{z'}{\sqrt{z'^2}}, \qquad e = \frac{z + z'}{\sqrt{(z + z')^2}}, \qquad e_0^2 = e_0'^2 = e^2 = 1. \tag{38}$$

The matrix $O[z', z]$ from (36) as well as every plane transformation is a product of two transformations of symmetry

$$O_0 = (I - 2e_2.e_2)(I - 2e_1.e_1), \qquad A_0 = (e_2.e_1 - e_1.e_2)/e_1 e_2 \tag{39}$$

and it corresponds to those unit vectors e_1 and e_2

$$e_1 = e_0, \qquad e_2 = e \qquad \text{or} \qquad e_1 = e, \qquad e_2 = e_0 \tag{40}$$

for which the equation (37) is fulfilled. The transformation of symmetry $I - 2e.e$ defines reflection according to a hyperplane orthogonal to the unit vector e and $(I - 2e.e)^2 = I$, $|I - 2e.e|^2 = 1$ are valid.
Matrix $O[z', z]$ in (36) may be found as one of the solutions of (37). The product of two plane transformations $O'' = O(A'') = O(A_0')O(A_0) = O[z'', \tilde{z}]O[z', z]$ has the structure of (39) when $\tilde{z} = z'$. Obviously, the case $\tilde{z} = z'$ is one of the most simple realization of the condition (31), which in the general case leads to the statement that between the four vectors $z, z', \tilde{z}, z''$, defining the simple matrices A_0 and A_0' in (35), not more than three are independent. That is why, if we consider the set of all simple matrices, every two of which satisfy (31), then the corresponding family of the plane rotations will be closed with respect to the operation (34) and defined in some subspace of three independent vectors of the n-dimensional space. This is just the situation in $\mathbb{R}^3$, where every transformation of the group of rotations SO(3) is plane. For example, all 3×3 rotation matrices in Euler angles are plane.
It is obvious, that the family of the plane transformations, defined through the set of all commutative simple matrices is given in the subspace of two independent vectors - in some plane. According to a theorem of Cartan [6], every transformation of the group SO(n) in n - dimensional vector space may be presented as a product of even numbers $\neq n$ transformations of symmetry. As far as the matrix $O[z', z]$ is a product of two transformations of symmetry, than any transformation of the group SO(n) may be presented as a product not greater than $[n/2]$ plane transformations. Naturally, this procedure has non-unique character. It has to be also noted that the theory of plane rotations given here relies heavily on the analogy with vector-parameterization of the rotation group SO(3), which we present on purpose in the section that follows.

VECTOR REPRESENTATION OF ROTATION MOTIONS

Let us consider the special orthogonal group SO(3) presenting the rotation motions

$$\mathsf{SO}(3) = \{O \in \mathrm{Mat}(3,\mathbb{R}) \ ; \ \det O = 1, \ \ OO^T = I\} \tag{41}$$

where $\mathrm{Mat}(3,\mathbb{R})$ is the group of 3×3 real matrices together with its Lie algebra (*i.e.*, its infinitesimal generators) consisting of the real skew-symmetrical 3×3 matrices. Again here is valid that if A belongs to the Lie algebra of SO(3), the matrix $I-A$ is invertible, and the Hamilton-Cayley transformation given in (2) is used. As an exception in the three-dimensional space, there exists a map (actually isomorphism) between vectors and skew-symmetric matrices, *i.e.*, if $c \in \mathbb{R}^3$, we have $c \to c^{\times}$, where $c^{\times}$ is the corresponding skew–symmetric matrix. Then we may write the SO(3) matrix in the form [9]

$$O = O(c) = (I+c^{\times})(I-c^{\times})^{-1} = \frac{(1-c^2)I + 2\,c\otimes c + 2\,c^{\times}}{1+c^2} \tag{42}$$

and consider it as a mapping from $\mathbb{R}^3$ to SO(3) for which the smooth inverse is

$$c^{\times} = \frac{O-O^T}{1+\mathrm{tr}(O)}\,. \tag{43}$$

Here I is the 3×3 identity matrix, $c\otimes c$ means diadic, $\mathrm{tr}(O)$ is the trace of the matrix O and "T" is the symbol for transposition of a matrix. The formula above provides us with an explicit parameterization of SO(3). The vector c is called a **vector-parameter**. It is parallel to the axis of rotation and its module $\|c\|$ is equal to $\tan(\alpha/2)$, where α is the angle of rotation. The so defined vector-parameters form a Lie group with the following composition law

$$c' = \langle c_1, c_2\rangle = \frac{c_1 + c_2 + c_1\times c_2}{1-c_1.c_2}\,. \tag{44}$$

The symbol "$\times$" means cross product of vectors. Every component of c can take all values from $-\infty$ to $+\infty$ without any restrictions, which is a great advantage compared with the obvious asymmetry in the Eulerian parameterization. The vector $c \equiv 0$ corresponds to the identity matrix $O(0)\equiv I$ and $-c$ produces the inverse rotation $O(-c)\equiv O^{-1}(c)$. Conjugating with elements from the SO(3) group leads to linear transformations in the vector-parameter space

$$O(c)\,O(c')\,O^{-1}(c) = O(c'')$$

where $c'' = O(c)\,c' = O_c\,c'$. Such a parameterization in the Lie group theory is called natural. It is worth mentioning also that no other parameterization possesses neither this property nor a manageable superposition law. This parameterization of SO(3) is known also as Gibbs' vector or Rodrigues' vector [13]. Some authors call it vector of finite rotations. Vector representation of rotations in three-dimensional space $\mathbb{R}^3$ is a subject of considerations of many authors, but we are the first in the literature using this parameterization as a Lie group with its nice group properties [9]. As for considering rotation problems of a rigid body and spacecrafts, it is used later also in modeling and control of open-loop mechanical systems like manipulators, vehicle devices, biomechanical systems [10]. Important properties of the composition law of the vector-parameter group are

$$\begin{aligned}
O(c')\,O(c) \;&=\; O(c''), \qquad c'' = \langle c',\, c\rangle = \frac{c'+c+c'\times c}{1-c'.c}\\
\langle c,\,0\rangle \;&=\; \langle 0,\,c\rangle = c, \qquad \langle c,\,-c\rangle = 0, \qquad \langle c',\,c\rangle \neq \langle c,\,c'\rangle\\
\langle\langle a,\,b\rangle,\,c\rangle \;&=\; \langle a,\,\langle b,\,c\rangle\rangle = \langle a,\,b,\,c\rangle, \qquad (O_a\,O_b)\,O_c = O_a(O_b\,O_c)\\
-\langle a,\,b\rangle \;&=\; \langle -b,\,-a\rangle, \qquad -\langle a,\,b,\,c\rangle = \langle c,\,-b,\,-a\rangle\\
O^{-1}(\langle a,b\rangle) \;&=\; (O(a)\,O(b))^{-1} = O^{-1}(b)\,O^{-1}(a) = O(-b)\,O(-a) = O(\langle -b,\,-a\rangle).
\end{aligned} \tag{45}$$

From the alternative expression of Cayley formula

$$O = (I-A)(I+A)^{-1} \tag{46}$$

we obtain the composition law in the case when c_1 is the first rotation

$$c' = \langle c_2, c_1 \rangle = \frac{c_1 + c_2 - c_1 \times c_2}{1 - c_1.c_2}. \tag{47}$$

Generally said, we have

$$c'_+ = \frac{c_1 + c_2 + c_1 \times c_2}{1 - c_1.c_2}, \qquad c'_- = \frac{c_1 + c_2 - c_1 \times c_2}{1 - c_1.c_2}.$$

The both vector-parameters c_+ and c_- correspond to the composition of two vectors in inverse order, that are symmetrically situated according to the plane defined by c_1 and c_2. This part is a proof for the strong analogy of plane orthogonal transformations in n - dimensional space with the vector-parameterization of the group SO(3).

ANALYTICAL FORM OF N × N ROTATION MATRIX

A block diagonal Givens matrix $R_i(\phi) \in \mathbb{R}^{n \times n}$ has the form [8]

$$R_i(\phi) = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & (\cos(\phi))_{i,i} & (-\sin(\phi))_{i,i+1} & 0 \\ 0 & (\sin(\phi))_{i+1,i} & (\cos(\phi))_{i+1,i+1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}, \qquad 0 \le \phi \le 2\pi \tag{48}$$

for $i \in 1, 2, \ldots, n-1$. As can be seen, the Givens matrix $R_i(\phi)$ involves only two coordinates that are affected by the rotation angle ϕ whereas the other directions, which correspond to eigenvalues 1, are unaffected by the rotation matrix. In dimension n there are $n-1$ Givens rotation matrices of the type (48). Composed they can generate a $n \times n$ matrix $R(\phi)$ according to

$$R(\phi) = R_1(\phi) R_2(\phi) \ldots R_{n-1}(\phi). \tag{49}$$

It is clear that the choice of the matrix $R(\phi)$ is a special one as the angles in matrices $R_i(\phi)$ are chosen to be equal. The explicit representation of $R(\phi)$ is of the form

$$\begin{bmatrix} \cos(\phi) & -\cos(\phi)\sin(\phi) & \ldots & \ldots & (-1)^n \cos(\phi)\sin^{n-2}(\phi) & (-1)^{n+1}\sin^{n-1}(\phi) \\ \sin\phi & \cos^2(\phi) & -\cos^2(\phi)\sin(\phi) & \ldots & (-1)^{n-1}\cos^2(\phi)\sin^{n-3}(\phi) & (-1)^n \cos(\phi)\sin^{n-2}(\phi) \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\cos^2(\phi)\sin(\phi) & \vdots \\ \vdots & & \ddots & \ddots & \cos^2(\phi) & -\cos(\phi)\sin(\phi) \\ 0 & \ldots & \ldots & 0 & \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{50}$$

The matrix (48) is almost an upper triangular matrix but with $\sin(\phi)$ on the first subdiagonal and the $(n-2) \times (n-2)$ submatrix starting at position $(2,2)$ is a Toeplitz upper triangular matrix. (A Toeplitz matrix T or diagonal - constant matrix, is a matrix in which each descending diagonal from left to right is constant, i.e., $T_{i,j} = T_{i+1,j+1}$).
Composed the Givens rotations can transform the basis of the space to any other frame in the space. The matrix $R(\phi)$ fulfills the properties $\det R(\phi) = 1$ and $R(\phi)R^T(\phi) = I_n$, and the property $R(\phi = 0) = I_n$ holds. When n is odd, the matrix $R(\phi)$ will have an eigenvalue 1 and the remaining eigenvalues are pairs of complex conjugates, whose product is 1. The last is valid also when n is even. Consequently, the matrix $R(\phi)$ is a rotation matrix and obviously orthogonal. We may conclude that every rotation matrix when expressed in a suitable coordinate systems, partitions into independent rotations of two-dimensional subspace like in (48).

CONCLUSION

The present paper is an interplay of the theory of SO(n) Lie group, the plane representations of any SO(n) element and the analogy with the vector-parameterization of the rotation group in the three-dimensional space. This study is provoked from the fact that the group parameterizations of the rotational motions in higher dimensions and their after–effects are of a great interest nowadays because of many applications in different scientific areas.

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