Analytic Description of the Equilibrium Shapes of Elastic Rings Under Uniform Hydrostatic Pressure

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Abstract. The parametric equations of the plane curves determining the equilibrium shapes that a uniform inextensible elastic ring could take subject to a uniform hydrostatic pressure are presented in an explicit analytic form. The determination of the equilibrium shape of such a structure corresponding to a given pressure is reduced to the solution of two transcendental equations. The shapes with points of contact and the corresponding (contact) pressures are determined by the solutions of three transcendental equations. The analytical results presented here confirm many of the previous numerical results on this subject but the results concerning the shapes with lines of contact reported up to now are revised.

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INTRODUCTION

The present paper addresses the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniformly distributed external force that acts normally to the ring in the ring plane.

Maurice Lévy [22] was the first who stated and studied the problem under consideration and reduced the determination of the foregoing equilibrium shapes in polar coordinates to two elliptic integrals for the arclength and polar angle regarded as functions of the squared radial coordinate. He found also several remarkable properties of the equilibrium ring shapes and concluded that if the pressure p is such that $p < (9/4)(D/\rho^3)$, where D and ρ are the ring bending rigidity and radius of the undeformed shape, respectively, then the ring possesses only the circular equilibrium shape.

Later on, Halphen [17] and Greenhill [15] derived exact solutions to this problem in terms of the Weierstrass elliptic functions on the ground of complicated analyses of the properties of the aforementioned elliptic integrals. Halphen (see [17, p. 235]) found out that non-circular shapes with $n \geq 2$ axes of symmetry are possible only for pressures greater than $p_n = (n^2 - 1)(D/\rho^3)$. Halphen [17] and Greenhill [15] presented also several examples of non-circular equilibrium ring shapes. It should be noted, however, that the exact solutions reported in [17, 15], representing the polar angle as a function of the radius, appeared to be intractable and many researchers continued searching exact solutions [4–10], while others used various approximations [24, 13] on the way

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to determine the equilibrium shapes of the ring.

Carrier [6] was the first who reconsidered the foregoing problem for the buckling of an elastic ring about half a century after the works by Lévy, Halphen and Greenhill. He expressed the curvature of the deformed ring in terms of the Jacobi cosine function [14–17] involving several unknown parameters to be determined by a system of algebraic equations. However, he succeeded to find approximate solutions to this system only for small deflections from the undeformed circular ring shape (see the analysis provided recently by Adams [1] who has criticized and developed Carrier's work [6]).

Tadjbakhsh and Odeh [24] studied the boundary-value problem describing the buckled shapes of the ring and the associated variational problem. They proved the existence of solutions to the boundary-value problem in the case of small deflections from the undeformed state and the existence of solutions of the associated variational problem (week solutions to the foregoing boundary-value problem) describing buckled shapes of an arbitrary deflection.

Watanabe and Takagi [27] thoroughly analyzed the variational problem for determination of the ring shapes stated by Tadjbakhsh and Odeh [24] and obtained analytic expressions for the curvature of the ring (in terms of the Jacobi elliptic functions and Carrier's parameters [6]) and formulas for the slope angle at the points where the curvature has extrema. They also proved that non-circular shapes with $n \geq 2$ axes of symmetry exist for pressures greater than $p_n = (n^2 - 1)(D/\rho^3)$, thus extending Halphen's result, and found out, moreover, that each such shape in unique. This paper completes the branch of analysis of the considered problem that is based more or less on the approach suggested more than half a century ago by Carrier [6].

In the recent papers [8–11], the present authors have studied the cylindrical equilibrium shapes of lipid bilayer membranes and the governing equation in the case of cylindrical shapes coincides with the differential equation for the curvature of the ring. The determination of the analytic solutions of this equation reported in [26] does not follow Carrier's approach [6]. Instead of this, the explicit formulas for the curvature of the buckled shapes are obtained in forms similar to those suggested by Zhang [28] for lipid bilayer membranes and by Fukumoto [14] in the context of fluid mechanics. In [26], the parametric equations of the directrices of the considered cylindrical surfaces are expressed in an explicit form, the necessary and sufficient conditions for such a surface to be closed are derived and several sufficient conditions for its directrix to be simple or self-intersecting are given.

The equilibrium shapes of closed planar elastic loops subject to the constraints of fixed length and enclosed area are studied also in the works by Arreaga *et al.* [3], Capovilla *et al.* [5] and Guven [16].

Flaherty *et al.* [13] presented a numerical determination of the equilibrium shapes of elastic rings subject to uniform external pressure. They suggested a scenario for the evolution of the equilibrium shape as the pressure increases, however, some stages of this scenario are not confirmed here.

The aim of the present study is to provide an analytic description of the equilibrium states of an elastic ring subject to a uniform hydrostatic pressure going as far as possible, to develop efficient computational procedures completing this analytic description and to re-examine some of the most important results obtained previously.

DIFFERENTIAL EQUATIONS FOR THE EQUILIBRIUM STATES

Let us consider a ring made of a homogeneous isotropic linearly elastic material and assume that it is represented by its middle axis. Suppose now that the ring is subject to a uniform external pressure p acting along the normal vector to its stress-free configuration. The analysis carried out in the present work is based on the following three assumptions: (*i*) the ring axis is inextensible; (*ii*) the pressure p preserves its magnitude and always acts as an external uniformly distributed force along the inward normal vector to the deformed ring axis, i.e., it is a uniform (simple) hydrostatic pressure; (*iii*) the deformed ring axis is a regular closed plane curve Γ parametrized by its arclength s.

Next, let the curve Γ be given by means of the coordinates $x(s)$, $z(s)$ of its position vector $\mathbf{r}(s)$ with respect to a certain rectangular Cartesian coordinate frame in the Euclidean plane, i.e., $\mathbf{r}(s) = x(s)\,\mathbf{i} + z(s)\,\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along the coordinate axes OX and OZ, respectively. Consequently, the unit tangent vector $\mathbf{t}(s)$ and the unit inward normal vector $\mathbf{n}(s)$ to the curve Γ are given as follows

$$\mathbf{t}(s) = x'(s)\,\mathbf{i} + z'(s)\,\mathbf{j}, \qquad \mathbf{n}(s) = -z'(s)\,\mathbf{i} + x'(s)\,\mathbf{j}. \tag{1}$$

Here and throughout this paper primes denote derivatives with respect to the arclength s. Recall that the foregoing unit tangent and normal vectors are related to the curvature $\kappa(s)$ of the curve Γ through the Frenet-Serret formulas [8]

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \qquad \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s). \tag{2}$$

Finally, let $M(s)$, $N(s)$ and $Q(s)$ denote the bending moment and the components of the stress resultant force $\mathbf{F}(s)$ along the tangent and normal vectors to the curve Γ, i.e.,

$$\mathbf{F}(s) = N(s)\mathbf{t}(s) + Q(s)\mathbf{n}(s). \tag{3}$$

Then, the particular constitutive equation relating the moment $M(s)$ with the curvature $\kappa(s)$ and the form of the stress-free configuration of the ring together with the system of differential equations

$$\mathbf{F}'(s) = -p\,\mathbf{n}(s), \qquad M'(s) = -\mathbf{F}(s)\cdot\mathbf{n}(s) \tag{4}$$

representing the local balances of the force and moment, respectively, in accordance with the assumption (*ii*), and the closure conditions following from the assumption (*iii*), which, without loss of generality, may be written in the form

$$\mathbf{r}(L) = \mathbf{r}(0), \qquad \mathbf{t}(L) = \mathbf{t}(0) \tag{5}$$

where L is the length of the deformed ring, determine entirely the equilibrium state of the ring under consideration (see, e.g., [2, Ch. 4)]). Here and throughout this paper the dot stands for dot (scalar) product of two vectors. Let us remark also that using equation (3) and the Frenet-Serret formulas (2) one can represent the system of differential equations (4) in the scalar form

$$N'(s) = Q(s)\kappa(s) \tag{6}$$

$$Q'(s) = -N(s)\kappa(s) - p \tag{7}$$

$$M'(s) = -Q(s). \tag{8}$$

PARAMETRIC EQUATIONS FOR THE EQUILIBRIUM SHAPES

The aim of this Section is to prove the most important facts concerning the problem under consideration established by Maurice Lévy in his memoir [22] and to derive the parametric equations of the equilibrium ring shapes.

Using the expression for the normal vector, see equations (1), one can integrate the equilibrium condition (4) and express the force vector in the form

$$\mathbf{F}(s) = pz(s)\,\mathbf{i} - px(s)\,\mathbf{j} \tag{9}$$

omitting the constant of integration since it is always possible to eliminate it by choosing the origin of the coordinate frame at a certain privileged point which, actually, is the one that Lévy called “centre of the elastic forces”, cf. [22, 1° (p. 9)]. equation (9) implies that at each point of the deformed ring configuration the magnitude of the force vector $F(s) = |\mathbf{F}(s)|$ is proportional to the magnitude of the position vector $r(s) = |\mathbf{r}(s)|$, the magnitude of the pressure $|p|$ being the coefficient of proportionality (cf. [22, 2° (p. 9)]), i.e.,

$$F(s) = |p|\, r(s). \tag{10}$$

Next, taking the dot product of both sides of equation (9) with the normal vector one gets, bearing in mind the second one of equations (1), the relation

$$\mathbf{F}(s)\cdot\mathbf{n}(s) = -p[x(s)x'(s) + z(s)z'(s)]$$

which allows to integrate the balance of moment equation (4_2) and to obtain

$$M(s) = \frac{p}{2}(r^2(s) + c) \tag{11}$$

where c is an arbitrary constant of integration, cf. [22, 3° (p. 9)]. It is noteworthy that the relations (9) – (11) hold regardless of the particular material properties of the ring and the form of its stress-free configuration.

In terms of the slope angle $\varphi(s)$ one has the expressions

$$x'(s) = \cos\varphi(s), \qquad z'(s) = \sin\varphi(s) \tag{12}$$

$$\kappa(s) = \varphi'(s) \tag{13}$$

and using equations (12) can rewrite equations (1) in the form

$$\mathbf{t}(s) \;=\; \cos\varphi(s)\,\mathbf{i} + \sin\varphi(s)\,\mathbf{j} \tag{14}$$

$$\mathbf{n}(s) \;=\; -\sin\varphi(s)\,\mathbf{i} + \cos\varphi(s)\,\mathbf{j}. \tag{15}$$

Then, combining equations (3), (9), (14) and (15) one obtains the parametric equations of the deformed ring shape in the form

$$\begin{aligned} x(s) &= -\frac{1}{p}Q(s)\cos\varphi(s) - \frac{1}{p}N(s)\sin\varphi(s) \\ z(s) &= -\frac{1}{p}Q(s)\sin\varphi(s) + \frac{1}{p}N(s)\cos\varphi(s). \end{aligned} \tag{16}$$

Evidently, in view of equation (14), the second one of the closure conditions (5) implies that the rotation number of the deformed ring axis is $2m\pi$, where m is an integer, i.e.,

$$\varphi(L) = \varphi(0) + 2m\pi \tag{17}$$

whereas the first one of them transforms, on account of equations (11), (17) and (17), into the obvious conditions for periodicity of the forces and moment

$$N(L) = N(0), \qquad Q(L) = Q(0), \qquad M(L) = M(0). \tag{18}$$

Again, neither the form of the parametric equations (17) nor the forms of the boundary conditions (17) and (18) depend on the particular material properties or the stress-free configuration of the ring.

Let us now assume that the constitutive equation of the ring is

$$M(s) = D(\kappa(s) - \kappa^{\circ}) \tag{19}$$

where D is its bending rigidity and $\kappa^{\circ} = 1/\rho$ is the curvature of its stress-free configuration, which is supposed to be a circle of radius ρ. Using equation (19) and the Frenet-Serret formulas (2) we can immediately integrate the system of differential equations (6) – (8) obtaining the following expressions for the tangent and normal components of the stress resultant force $\mathbf{F}(s)$

$$N(s) = -\frac{D}{2}(\kappa^2(s) - 2\mu), \qquad Q(s) = -D\kappa'(s) \tag{20}$$

and the single ordinary differential equation for the ring curvature

$$\kappa''(s) + \frac{1}{2}\kappa^3(s) - \mu\kappa(s) - \sigma = 0 \tag{21}$$

where $\sigma = p/D$ and μ is an arbitrary constant of integration. On the other hand, combining equations (10), (11) and (19) we obtain the relation

$$N^2(s) + Q^2(s) = 2pD(\kappa(s) - \kappa^{\circ}) - p^2 c$$

which, in view of equations (20), implies

$$\kappa'(s)^2 = P(\kappa(s)) \tag{22}$$

where $P(\kappa)$ is a fourth-order polynomial of the curvature κ of the form

$$P(\kappa) = -\frac{1}{4}\kappa^4 + \mu\kappa^2 + 2\sigma\kappa + \varepsilon \tag{23}$$

whose free term $\varepsilon = -2\sigma\kappa^{\circ} - \sigma^2 c - \mu^2$ incorporates all constants of the integrations introduced so far. Actually, equation (22) is a first integral of equation (21) (see [26, Section 2] for more details). In this context, ε is viewed as an arbitrary constant of integration.

Each sufficiently smooth real-valued solution $\kappa(s)$ of an equation of form (22) corresponding to a certain triple of given values of the parameters μ, ε and $\sigma \neq 0$ generates, up to a rigid motion in the plane, a unique plane curve Γ of curvature $\kappa(s)$. The components of the position vector of this curve can by expressed in the form

$$
\begin{aligned}
x(s) &= \frac{1}{\sigma}\kappa'(s)\cos\varphi(s) + \frac{1}{2\sigma}(\kappa^2(s) - 2\mu)\sin\varphi(s) \\
z(s) &= \frac{1}{\sigma}\kappa'(s)\sin\varphi(s) - \frac{1}{2\sigma}(\kappa^2(s) - 2\mu)\cos\varphi(s)
\end{aligned}
\tag{24}
$$

obtained by substituting equations (20) in the general formulas (17). However, the parametric equations (25) describe a shape that a ring of bending rigidity D could take being subject to pressure $p = \sigma D$ if and only if the regarded solution $\kappa(s)$ of the respective equation of form (22) is such that the closure conditions (17) and (18) hold for $L = 2\pi\rho$; note that the latter equality follows from the assumption (i). If this is the case, then the respective solution $\kappa(s)$, its first derivative $\kappa'(s)$ and its indefinite integral

$$
\varphi(s) = \int \kappa(s)\mathrm{d}s \tag{25}
$$

cf. equation (13), determine entirely the equilibrium state of the pressurized ring. Indeed, the shape of the ring is determined explicitly by equations (25) and the values of the moment and forces acting along the ring axes are given by the formulas (19) and (20), respectively.

It should be remarked that each such solution $\kappa(s)$ is necessarily a periodic function with period $L = 2\pi\rho$, due to the condition (18), and if T is its least period, then $L = nT$, where n is a positive integer. Since $\varphi(nT) = n\varphi(T)$, as follows by formula (25), the closure condition (17) takes the form

$$
\varphi(T) = \frac{1}{n}\varphi(0) + \frac{2m\pi}{n}\cdot \tag{26}
$$

DETERMINATION OF THE EQUILIBRIUM SHAPES

In the present study, our primary interest is in the determination of the equilibrium ring shapes that are curves without intersections, i.e., simple curves. Assuming $\varphi(0) = 0$, the closure condition (26) for such a curve reads

$$
\varphi(T) = \pm\frac{2\pi}{n} \tag{27}
$$

with $n \geq 2$ due to the four vertex theorem (see, e.g., [9]) and $m = \pm 1$ since the rotation number of a simple regular closed curve must be $\pm 2\pi$, see [19]. Note, however, that there exist regular closed curve with rotation number $\pm 2\pi$, which are not simple.

Then, substituting the expression $T = (4/\lambda)K(k)$ for the least period of t he respective solutions of equation (22) in the general formula [26, formula (23)] for the corresponding slope angle $\varphi(s)$ one can rewrite the closure condition (27) in the form

$$\frac{(A+B)(\alpha-\beta)}{2\lambda(A-B)}\Pi(-C,k)+\frac{A\beta-B\alpha}{\lambda(A-B)}K(k)=\pm\frac{\pi}{2n} \tag{28}$$

where

$$\alpha, \qquad \beta, \qquad \gamma=-\frac{\alpha+\beta}{2}+\mathrm{i}\eta, \qquad \delta=-\frac{\alpha+\beta}{2}-\mathrm{i}\eta, \qquad (\alpha<\beta)$$

are the roots of the polynomial (23) and

$$A=\sqrt{4\eta^2+(3\alpha+\beta)^2}, \qquad B=\sqrt{4\eta^2+(\alpha+3\beta)^2}$$

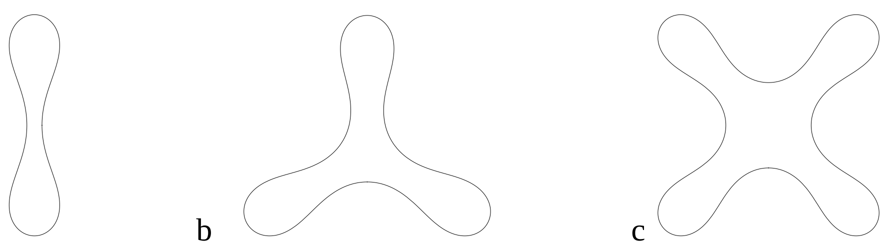
$$\lambda=\frac{1}{4}\sqrt{AB}, \qquad k=\sqrt{\frac{1}{2}-\frac{4\eta^2+(3\alpha+\beta)(\alpha+3\beta)}{2AB}}\cdot$$

Finally, substituting the same expression for the period T in the relation $L=nT$ in order to take into account that the length of the ring L is fixed and does not change upon deformation, see assumption (i), one obtains

$$\frac{1}{\lambda}K(k)=\frac{\pi}{2n}\cdot \tag{29}$$

after setting for simplicity, without loss of generality, $L=2\pi$, i.e., $\kappa^\circ=\rho=1$.

The left-hand sides of equations (28) and (29) can be expressed as functions of the positive parameters σ, η and $q=\beta-\alpha$. Thus, given an integer $n\geq 2$ and a pressure p by means of the parameter σ (called hereafter simply “pressure”) the problem for the determination of the foregoing equilibrium shapes of the ring corresponding to this pressure is reduced to the computation of the values of η and q from the transcendental equations (28) and (29).



a b c

FIGURE 1. Equilibrium ring shapes corresponding to: a) $\sigma=4.75$ (2-fold symmetry); b) $\sigma=16.25$ (3-fold symmetry); c) $\sigma=35.25$ (4-fold symmetry).

It is important to notice that this problem has no nontrivial solution if $0<\sigma\leq\sigma_{bn}$ and has a unique nontrivial solution if $\sigma>\sigma_{bn}$, see [27, Theorem 2]. Here, $\sigma_{bn}=n^2-1$ is the so-called buckling pressure and by a trivial solution we mean the one, which corresponds to the ring shape that is a circle of radius $\rho=1$.

Given an integer $n\geq 2$ and a pressure $\sigma>\sigma_{bn}$, the transcendental equations (28) and (29) are solved numerically in two steps using `Mathematica`®. First, the two curves in the (η, q) plane defined by equations (28) and (29) are plotted using the routine `ContourPlot` in order to identify roughly the values of the coordinates of their intersection point. Then, these values are put as starting values in the routine `FindRoot`, which is employed to obtain the solutions η and q of the system of equations (28), (29) with a sufficient accuracy. Once such a solution is determined, formulas [12, (38), (39), (40), (44)] and the parametric equations (25) allow to depict the corresponding equilibrium ring shapes using the routine `ParametricPlot`. Three examples of such shapes, which confirm the results presented in [13] are given in Fig. 1.

EQUILIBRIUM SHAPES WITH POINTS OF CONTACT

It is established in [24, 13] that for each mode $n = 2,3,4$ there is a value of the pressure σ, called contact pressure and denoted by σ_{cn}, at which some points of the respective buckled ring shape of n-fold symmetry come into contact. In the aforementioned works, it is also observed that if the applied pressure σ is such that $\sigma_{bn} < \sigma < \sigma_{cn}$, then the corresponding buckled shape of n mode is simple. It should be noted, that the values for the contact pressures reported in [24, 13] are obtained solving numerically a rather complicated nonlinear boundary-value problem.

In the present study, the determination of the non-circular equilibrium ring shapes with points of contact and the respective contact pressures is reexamined being reduced to the computation of the common solutions of the transcendental equations (28), (29) and one more transcendental equation in the way described below.

Proceeding to the examination of this problem, let us first clarify, slightly extending the definition used in [13], that an n-mode equilibrium ring shape Γ_n is said to have a point of contact if it is not self-intersecting, but there is at least one couple of values s_1 and s_2 of the arclength s such that $0 < s_1 < s_2 < L$ and

$$\mathbf{r}(s_2) = \mathbf{r}(s_1), \qquad \mathbf{t}(s_2) = -\mathbf{t}(s_1). \tag{30}$$

This means that at the point of contact $\mathbf{r}(s_2) = \mathbf{r}(s_1)$ the curve Γ_n is tangent to itself. Such a double point on a curve is called a *cusp* or *tacnode*, see [23]. The objective now is to reformulate the above conditions in a form suitable for the developing of an efficient procedure for computation of the contact pressures corresponding to the foregoing equilibrium ring shapes.

For that purpose, it is convenient to use the relations

$$\kappa(s) = \frac{\sigma}{2} r^2(s) - \frac{\mu^2 + \varepsilon}{2\sigma}$$

and

$$\mathbf{r}(s) \cdot \mathbf{t}(s) = \frac{1}{\sigma} \kappa'(s), \qquad \mathbf{r}(s) \cdot \mathbf{n}(s) = -\frac{1}{2\sigma}(\kappa^2(s) - 2\mu)$$

which follow from equations (22) – (25) and allow, taking into account equations (14) and (15), conditions (30) to be cast in the form

$$\kappa(s_2) = \kappa(s_1), \qquad \kappa'(s_2) = -\kappa'(s_1) \tag{31}$$

$$\kappa^2(s_2) - 2\mu = \kappa^2(s_1) - 2\mu = 0 \tag{32}$$

$$\varphi(s_2) = \varphi(s_1) + (2l+1)\pi \tag{33}$$

where l is an integer.

It is shown in [12] that conditions (31) – (33) take the form

$$\varphi(s_1^-) = \frac{\pi}{n} - \frac{\pi}{2}, \qquad \varphi(s_1^+) = -\frac{\pi}{2} \tag{34}$$

where

$$s_1^{\pm} = \frac{1}{\lambda} F\left(\arccos \frac{A\beta + B\alpha \pm \sqrt{2\mu}(A+B)}{A\beta - B\alpha \pm \sqrt{2\mu}(A-B)}, k\right) \tag{35}$$

are the two solutions of the equation $\kappa^2(s) = 2\mu$ that belong to the interval $[0, T/2]$, and $F(\cdot,\cdot)$ is the incomplete elliptic integral of the first kind.

Thus, two triples of transcendental equations (28), (29) and $(34)_1$ or $(34)_2$ arise for the determination of the n-mode equilibrium ring shapes with points of contact. In both cases, the respective transcendental equations involve as unknowns only the four parameters σ, η, q and n since the arclengths s_1^- and s_1^+, which may correspond to points of contact are determined explicitly in terms of these parameters by formulas (35), and the same holds true for the left hand sides of equations $(34)_1$ and $(34)_2$ in view of the general expression [26, (23)] for the slope angle. Of course, one should remember that the sign of the right hand side of equation (27) is plus in this context.

To summarize: given an integer $n \geq 2$, each solution of any one of the aforementioned two triples of transcendental equations gives the value of the contact pressure σ_{cn} and the values of the parameters η and q determining in this way an equilibrium ring shape of n-fold symmetry with points of contact.

Solving numerically the foregoing two systems of transcendental equations using the routine `FindRoot` in `Mathematica`® we have found that the system consisting of equations (28), (29) and $(34)_2$ does not have solutions for $2 \leq n \leq 15$, but the system of equations (28), (29) and $(34)_1$ has a unique solution for each such mode n. Our conjecture is that this happens for all modes. The equilibrium ring shapes with points of contact corresponding to the contact pressures σ_{c2}, σ_{c3} and σ_{c4} are depicted in Fig. 2.

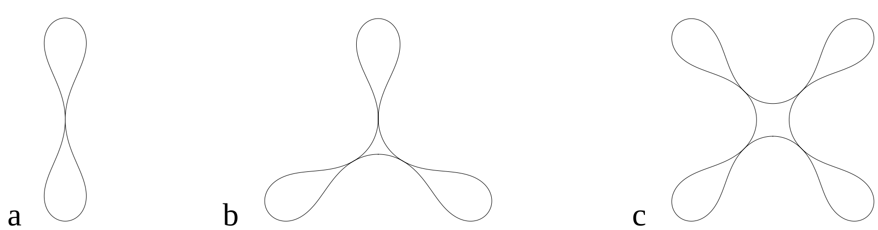


FIGURE 2. Ring shapes with points of contact: a) $\sigma = 5.247$; b) $\sigma = 21.65$; c) $\sigma = 51.844$.

The values of contact pressures σ_{cn} obtained for $n = 2, 3, 4$ confirm exactly the results presented in [13, formulas (2.11)], see also Fig. 2, but the latter are interpreted therein as the lowest values of the pressures at which an isolated point of contact occurs. In [13], it is claimed that for each mode n there exists a continuous range of pressures, from σ_{cn} up to a certain pressure denoted by σ_{0n}, such that for each $\sigma_{cn} \leq \sigma \leq \sigma_{0n}$ the respective ring shape exhibits contacts at isolated points only. The pressure σ_{0n} is set in [13] to be the one for which the curvature at the corresponding contact point is zero. If so, however, then in view of equations (32) $\mu = 0$ and $s_1^- = s_1^+ = s_0$. Therefore, the equation $\kappa(s) - 2\mu = 0$ has exactly one solution for $s \in [0, T_1/2]$ and hence, according to [26, Theorem 3], the corresponding ring shape is self-intersecting. Thus, in this respect the results of Flaherty *et al.* [13] turn out to be inaccurate in spite of the fact that they are widely accepted and even confirmed numerically by other authors (see, e.g., [4]).

Actually, for all modes n in the range $2 \le n \le 15$ our computations show that if the applied pressure σ is such that $\sigma_{bn} < \sigma < \sigma_{cn}$, then the corresponding buckled shape of n mode is simple, while for $\sigma > \sigma_{cn}$ this shape always has points of self-intersection. Our conjecture is that this behaviour is inherent to all modes. Let us recall that for each $\sigma > \sigma_{bn}$ the corresponding ring shape of n-fold symmetry is unique, see [27, Theorem 2].

EQUILIBRIUM SHAPES WITH LINES OF CONTACT

Apparently, for rings of finite thickness self-intersecting shapes are not possible because they are not planar, nevertheless for a very thin ring such a shape may be considered as a good approximation of its equilibrium state. There is a good reason to expect that rings of finite thickness subject to sufficiently high pressure posses equilibrium shapes with lines of contact.

Flaherty *et al.* [13] suggest similarity transformations to be used for the determination of such shapes, but realize this idea in a very complicated way. Moreover, the construction developed in [13] for the said purpose makes use of the curves Γ_{0n} corresponding to the pressures σ_{0n}, which are wrongly regarded as curves with isolated points of contact as it was noted and discussed above.

Below, an alternative approach is presented for constructing equilibrium ring shapes with lines of contact based on the same "similarity" idea that actually arises out of the following property of equations (21) and (22).

Under the transformation $(s, \kappa) \longmapsto (s/\tau, \tau\kappa)$, where τ is an arbitrary real number, each equation of form (21) corresponding to certain constants μ and σ transforms into an equation of the same form but with new coefficients: $\mu \longmapsto \tau^2\mu$, $\sigma \longmapsto \tau^3\sigma$. The same holds true for equation (22) if $\varepsilon \longmapsto \tau^4\varepsilon$ in addition. In other words, equations (21) and (22) are invariant with respect to the similarity transformation $\Lambda : (s, \kappa; \mu, \sigma, \varepsilon) \longmapsto (s/\tau, \tau\kappa; \tau^2\mu, \tau^3\sigma, \tau^4\varepsilon)$. Consequently, the parametric equations (25) imply that the shapes whose parameters are related by such a transformation Λ are similar, the respective scaling factor being $1/\tau$. Accordingly, if a closed curve Γ is scaled in this way, then its length L and area A change to L/τ and A/τ^2, respectively.

Thus, given $n \ge 2$, let the curve Γ_{cn} of length $L_{cn} = 2\pi$ be the equilibrium shape with points of contact corresponding to the contact pressure σ_{cn} and let $\hat{\Gamma}$ be the shape (of the same length) with lines of contact corresponding to a pressure $\hat{\sigma} > \sigma_{cn}$. The curve $\hat{\Gamma}$ is constructed in two steps. First, scaling the curve Γ_{cn} with a factor $(\hat{\sigma}/\sigma_{cn})^{1/3}$ one obtains another curve $\hat{\Gamma}_{cn}$ which has the same number of contact points because it is similar to the curve Γ_{cn} but corresponds to the pressure $\hat{\sigma}$ and its length is $\hat{L}_{cn} = 2\pi(\sigma_{cn}/\hat{\sigma})^{1/3} < L_{cn}$. Then, the curve $\hat{\Gamma}$ is obtained by substituting each point of contact of the curve $\hat{\Gamma}_{cn}$ by a line segment of length $2\pi(1 - (\sigma_{cn}/\hat{\sigma})^{1/3})/n$ along the respective symmetry axis of the curve $\hat{\Gamma}_{cn}$ so as its total length to become 2π. Examples of shapes with lines of contact are presented in Fig. 3 and Fig. 4.

It is clear that the tangent, normal and position vectors of a shape $\hat{\Gamma}$ with lines of contact constructed in the foregoing way are continuous at each point of the curve $\hat{\Gamma}$. However, its curvature suffers jumps at the end points of the line segments used to substitute the contact points of the respective auxiliary curve $\hat{\Gamma}_{cn}$ because the limit values of the curvature from the bent parts of the curve and from the line segments are $-\sqrt{2\mu} \neq 0$ and zero, respectively. Consequently, the moment and force also suffer jumps at the aforementioned points since their limit values from the bent parts of the curve are

$$M_b = -D\left(\sqrt{2\mu} + \kappa^{\circ}\right), \qquad N_b = 0, \qquad Q_b = \pm D\sqrt{P\left(-\sqrt{2\mu}\right)} \tag{36}$$

while along each line of contact the resultant pressure is zero and

$$M_l = -D\kappa^{\circ}, \qquad N_l = 0, \qquad Q_l = 0. \tag{37}$$

Equations (36) and (37) are consequences of the constitutive equation (19), the general solution (20) of equations (6) – (8) and equation (22).

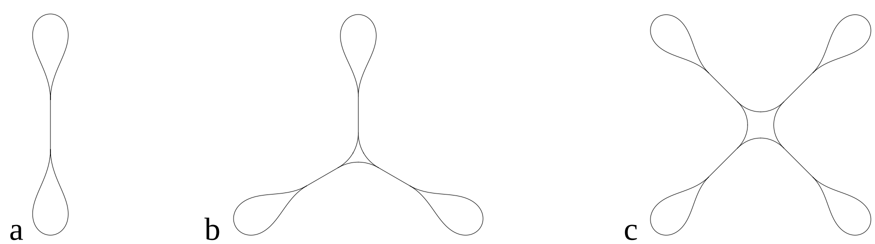


FIGURE 3. Shapes with lines of contact corresponding to: a) $\sigma = 10.34$ (2-fold symmetry); b) $\sigma = 81.81$ (3-fold symmetry); c) $\sigma = 207.2$ (4-fold symmetry).

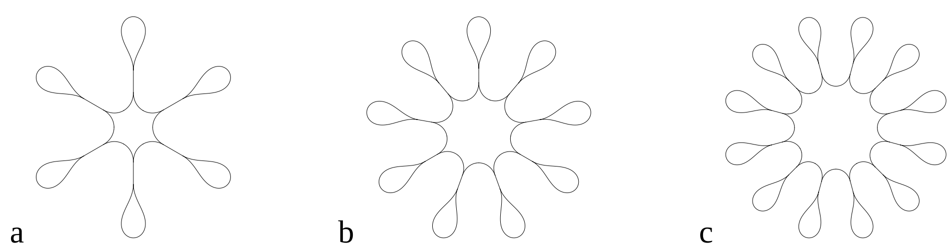


FIGURE 4. Shapes with lines of contact corresponding to: a) $\sigma = 400$ (6-fold symmetry); b) $\sigma = 800$ (9-fold symmetry); c) $\sigma = 1500$ (12-fold symmetry).

Thus, the local balances (4) of the force and moment are violated for the shapes with lines of contact. Fortunately, however, the total balances

$$\oint_{\hat{\Gamma}} \mathbf{F}'(s)\mathrm{d}s = -\oint_{\hat{\Gamma}} p\,\mathbf{n}(s)\mathrm{d}s, \qquad \oint_{\hat{\Gamma}} M'(s)\mathrm{d}s = -\oint_{\hat{\Gamma}} \mathbf{F}(s)\cdot\mathbf{n}(s)\mathrm{d}s \tag{38}$$

of these quantities are satisfied. Indeed, equations (38) hold on the curve $\hat{\Gamma}_{cn}$ since it corresponds to an equilibrium shape without jump discontinuities of the force and moment.

On the other hand, $\hat{\Gamma} = \hat{\Gamma}_{cn} \cup \{\text{Line segments}\}$ and the integrals in equations (38) taken along the line segments are equal to zero because $p = 0$, $M'(s) = 0$ and $\mathbf{F}(s) = \mathbf{0}$ there.

In our opinion, this property of the constructed curves $\hat{\Gamma}$ allows these shapes to be regarded as equilibrium ring shapes with lines of contact at least in the week sense discussed above.

CONCLUDING REMARKS

In the present study, the problem for determination of the equilibrium shapes of a circular inextensible elastic ring subject to a uniform hydrostatic pressure is reexamined. This problem was stated and studied by Maurice Lévy in his memoir [22] more than a century ago. Here, a concise derivation of the most important facts established in [22], see 1° – 3° (p. 9), concerning the existence of a "centre of the elastic forces", following by equation (9), and the properties reflected by equations (10) and (11) is given in Section 3. Then, the parametric equations of the equilibrium shapes are expressed through the forces and slope angle, see equations (17). It is noteworthy that neither the relations (9) – (11) nor the forms of the parametric equations (17) or boundary conditions (17) and (18) depend on the particular material properties or stress-free configuration of the ring.

Further, assuming that the stress-free configuration of the ring is a circle of radius ρ, the case of a linear constitutive equation of form (19) is considered. In this case, the equilibrium state of the ring is determined by the periodic solutions of the nonlinear ordinary differential equation (22) for the ring curvature, which meet the closure condition (26). In fact, the shape of the ring is given explicitly by the parametric equations (25) and the values of the moment and forces acting along the ring axes are obtained by equations (19) and (20), respectively, using the explicit analytic expressions for all periodic solutions of equation (22) and for the corresponding slope angles presented in [26].

With these results, the determination of the equilibrium shapes corresponding to a given pressure σ is reduced to the computation of the common solutions of two transcendental equations (28) and (29). It is also shown that the pressures at which the ring attains a shape with isolated points of contact can be obtained computing the common solutions of three transcendental equations (28), (29) and $(34)_1$. In contrast to the assertion in [13] that for each mode n there exists a range of pressures for which the respective ring shape has only isolated points of contact, we found, that for each mode $2 \leq n \leq 15$ there is a unique such pressure, namely σ_{cn}. Moreover, for all modes in the range $2 \leq n \leq 15$ our computations show that if the applied pressure σ is such that $\sigma_{bn} < \sigma < \sigma_{cn}$, then the corresponding buckled shape of n mode is simple, while for $\sigma > \sigma_{cn}$ this shape always has points of self-intersection. Our conjecture is that this behaviour is inherent to all modes.

Section 8 concerns the equilibrium ring shapes with lines of contact that are expected to occur for pressures greater than the respective contact pressure instead of the self-intersecting shapes (unnatural for planar rings) predicted by the considered model. Here, the construction of these shapes is based on the similarity properties of equations (22) and (25) following in general outline the idea suggested by Flaherty *et al.* in [13], but the uniqueness of the contact pressures σ_{cn} is taken into account. The shapes obtained in this way are shown to satisfy the total balances (38) of the respective forces and moments, which is a good reason to consider them as equilibrium shapes.

The interested reader can find the `Mathematica`® notebooks developed for the solution of the transcendental equations (28), (29) and $(34)_1$ as well as the notebooks developed for the construction of the shapes with lines of contact by visiting `http://www.bio21.bas.bg/ibf/dpb_files/mfiles/`.

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