On Some Deformations of the Cassinian Oval

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Abstract. The work is concerned with the determination of explicit parametric equations of several plane curves whose curvature depends solely on the distance from the origin. Here we suggest and exemplify a simple scheme for reconstruction of a plane curve if its curvature belongs to the above-mentioned class. Explicit parameterizations of generalized Cassinian ovals including also the trajectories of a charged particle in the field of a magnetic dipole are derived in terms of Jacobian elliptic functions and elliptic integrals.

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INTRODUCTION

Remarkably, the curvature of a lot of the famous plane curves (see [5, 14]), such as conic sections, Bernoulli's lemniscate, Cassinian ovals and many others, depends solely on the distance from a certain point in the Euclidean plane, which may be chosen as its origin.

The most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [3, p. 296] or [9, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that if the curvature is given by a function of its position the situation is even more complicated. Viewing the Frenet-Serret equations as a ficticious dynamical system it was proven in [11] that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [10] had already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [11] or mechanical [10] ones proposed in the aforementioned papers. The method is illustrated on a class of curves whose curvature depend solely on the distance from the origin.

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THE FRENET-SERRET EQUATIONS

If $x(s)$, $z(s)$ and $\theta(s)$ denote the Cartesian coordinates of a curve in the plane XOZ and the slope of the tangent to it with respect to the OX axis regarded as functions of the arc-length parameter s one has the following geometrical relations

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos\theta(s), \qquad \frac{\mathrm{d}z(s)}{\mathrm{d}s} = \sin\theta(s) \tag{1}$$

which can be deduced from the Frenet-Serret equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{T}, \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} \tag{2}$$

as well (see also Fig. 1) in which $\mathbf{x} = (x,z)$, $\mathbf{T}$ and $\mathbf{N}$ are respectively the position, tangent and normal vectors to the curve. Let us recall that

$$\mathbf{T} = (\cos\theta, \sin\theta), \qquad \mathbf{N} = (-\sin\theta, \cos\theta). \tag{3}$$

We will proceed (as suggested but not pursued in [10]) by going to the so-called co-moving frame $(\mathbf{T},\mathbf{N})$ associated with the curve in which

$$\mathbf{x} = \xi\mathbf{T} + \eta\mathbf{N}. \tag{4}$$

According to the Frenet-Serret equations (2), the components $\xi(s)$ and $\eta(s)$ of the position vector with respect to the co-moving frame meet the following equations

$$\dot{\xi} = \kappa\eta + 1, \qquad \dot{\eta} = -\kappa\xi. \tag{5}$$

Hereafter, the dot indicates differentiation with respect to the arc-length parameter s. Below, it is shown that in the cases when the curvature of a curve depends solely on the distance from the origin, equations (5) allow the components ξ and η of the position vector and the slope angle θ to be expressed by quadratures.

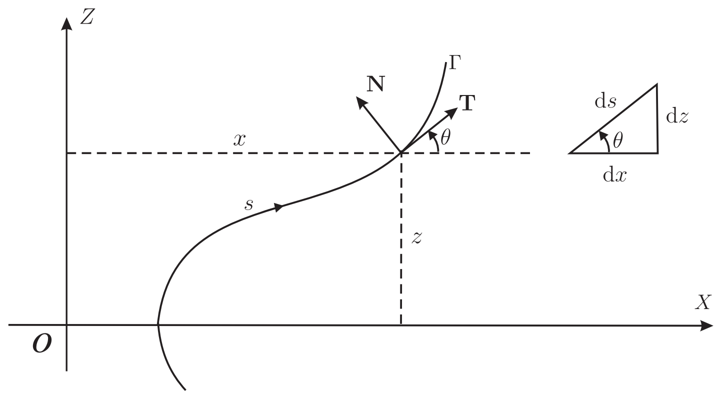

FIGURE 1. Geometry of the plane curve.

INTEGRATION

Multiplying the first equation in (5) by ξ, the second one by η and summing the so obtained expressions we find that

$$\xi = r\dot{r} \tag{6}$$

since

$$\xi^2 + \eta^2 = r^2 \tag{7}$$

as follows by equation (4), where $r = |\mathbf{x}| = \sqrt{x^2+z^2}$. Substituting expression (6) back into the second one of equations (5) and integrating we obtain

$$\eta = g(r) + c \tag{8}$$

where

$$g(r) = -\int r\kappa(r)\mathrm{d}r \tag{9}$$

and c is the integration constant. In view of equations (6) and (8), relation (7) leads to the following first-order ordinary differential equation

$$\dot{r}^2 = \frac{1}{r^2}\left[r^2 - (g(r)+c)^2\right] \tag{10}$$

for the radial coordinate r.

Thus, given explicitly the curvature κ of a plane curve as a function of the radial coordinate r, one can try to express the general solution of equation (10) in a suitable explicit form. If such an attempt is successful, then the components ξ and η of the position vector can be found explicitly by equations (6), (8) – (10), while the expression for the slope angle θ can be obtained by solving the integral in the right-hand-side of the relation

$$\theta(s) = \int \kappa(r(s))\mathrm{d}s \tag{11}$$

which is implied by the first of equations (1).

CASSINIAN OVALS

Let Γ be a plane curve given implicitly by the equation

$$F(x,z) = 0.$$

Then the curvature κ of the curve Γ is determined, see e.g. [6], by the well-known relation

$$\kappa(x,z) = \frac{|F_{xx}F_z^2 - 2F_{xz}F_xF_z + F_{zz}F_x^2|}{(F_x^2+F_z^2)^{3/2}}\,_{|F=0}. \tag{12}$$

A Cassinian Oval is a plane curve given by a quartic polynomial equation of the form

$$F(x,z) = \left[(x-a)^2 + z^2\right]\left[(x+a)^2 + z^2\right] - b^4 = 0 \tag{13}$$

where a and b are real numbers, see, e.g., [14]. Hence, according to formulas (12) and (13), the curvature of the Cassinian Oval has the form

$$\kappa = \frac{a^4 - b^4}{2b^2 r^3} + \frac{3r}{2b^2}\,. \tag{14}$$

GENERALIZED CASSINIAN OVALS

One can generalize the expression for the curvature of the Cassinian oval by writing it in the form

$$\kappa = \frac{\lambda}{r^3} + 3\mu r \tag{15}$$

in which λ and μ are assumed to be real constants. Next, one can try to find the respective parametric equations of the curves whose curvature is specified in (15) by using the approach described in the previous section. It is easy to be seen that strongly positive values of the parameters λ and μ reproduce exactly the Cassinian oval due to the relations

$$b^2 = \frac{1}{2\mu}, \qquad a^4 = \frac{4\lambda\mu + 1}{4\mu^2}\cdot \tag{16}$$

The above range of parameters could be easily extended by adding negative values of λ which fulfill together with μ the inequality

$$4\lambda\mu + 1 > 0. \tag{17}$$

However, for any other combination of their values one will get some curve which should be considered as a deformation of the parent curve. In the present paper this idea is traced in the following two cases: (I) $\lambda, \mu \neq 0$ and choosing the integration constant c in equation (10) to be zero, (II) $\mu = 0$, $c < 0$ and $0 < \lambda < c^2/4$.

In the first case, it is convenient to treat the problem in terms of the variable $\zeta = r^2$, which means, according to formulas (15), (6) and (8) – (10), that

$$\kappa = \frac{\lambda}{\zeta^{3/2}} + 3\mu\sqrt{\zeta}, \qquad g = \frac{\zeta^2\mu - \lambda}{\sqrt{\zeta}}, \qquad \xi = \frac{1}{2}\dot{\zeta}, \qquad \eta = g \tag{18}$$

and

$$\dot{\zeta}^2 = \frac{4\mu^2}{\zeta} R(\zeta) \tag{19}$$

where

$$R(\zeta) = -\left(\zeta^2 + \frac{1}{\mu}\zeta - \frac{\lambda}{\mu}\right)\left(\zeta^2 - \frac{1}{\mu}\zeta - \frac{\lambda}{\mu}\right).$$

In terms of a new variable τ such that

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{2\mu}{\sqrt{\zeta(\tau)}} \tag{20}$$

as a consequence of relations (20), (18), (19), and (11) we have

$$\xi(\tau) = \frac{\mu}{\sqrt{\zeta(\tau)}} \frac{\mathrm{d}\zeta(\tau)}{\mathrm{d}\tau}, \qquad \eta(\tau) = \frac{\lambda - \mu\zeta^2(\tau)}{\sqrt{\zeta(\tau)}} \tag{21}$$

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^2 = R(\zeta) \tag{22}$$

and

$$\theta(\tau) = \int \left(\frac{\lambda}{2\mu} \frac{1}{\zeta(\tau)} + \frac{3}{2}\zeta(\tau) \right) \mathrm{d}\tau. \tag{23}$$

Using the approach described in the Appendix one can obtain the general solution of equation (22) in the form

$$\zeta(\tau) = \alpha + \frac{(\delta-\alpha)(\gamma-\alpha)}{(\gamma-\alpha)+(\delta-\gamma)\,\mathrm{sn}^2(\omega\tau,k)} \tag{24}$$

where $\alpha < \beta < \gamma < \delta$ are the roots of the polynomial $R(\zeta)$ and

$$k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)}}, \qquad \omega = \frac{1}{2}\sqrt{(\gamma-\alpha)(\delta-\beta)}.$$

Evidently, the roots of the polynomial $R(\zeta)$ are

$$\pm\frac{1}{2\mu}\left(\sqrt{1+4\mu\lambda}+1\right), \qquad \pm\frac{1}{2\mu}\left(\sqrt{1+4\mu\lambda}-1\right).$$

They should be real, otherwise equation (22) does not have real solutions, which implies the restriction (17) on the values of the parameters μ and λ. Depending on the signs of the foregoing parameters, one can denote properly each of the above expressions so as $\alpha < \beta < \gamma < \delta$. For instance, when $\mu > 0$ and $\lambda > 0$ the roots can be chosen as follows

$$\alpha = -\delta = -\frac{1}{2\mu}\left(\sqrt{1+4\mu\lambda}+1\right), \qquad \beta = -\gamma = -\frac{1}{2\mu}\left(\sqrt{1+4\mu\lambda}-1\right).$$

Solving the integral in the right hand side of equation (23) one obtains the explicit expression

$$\theta(\tau) = \frac{3\alpha^2\mu+\lambda}{2\alpha\mu}\tau + \frac{\lambda(\alpha-\delta)}{2\alpha\delta\mu\omega}\Pi_1(\tau) - \frac{3(\alpha-\delta)}{2\omega}\Pi_2(\tau) \tag{25}$$

for the slope angle, where

$$\Pi_1(\tau) = \Pi\left(\frac{\alpha(\delta-\gamma)}{\delta(\alpha-\gamma)}, \mathrm{am}(\tau\omega,k), k\right), \qquad \Pi_2(\tau) = \Pi\left(\frac{\delta-\gamma}{\alpha-\gamma}, \mathrm{am}(\tau\omega,k), k\right).$$

Here, $\Pi(\cdot,\cdot,\cdot)$ and $\mathrm{am}(\cdot,\cdot)$ denote the incomplete elliptic integral of the third kind and respectively the Jacobi amplitude (more details on the subject of elliptic functions and integrals can be found in [1, 8] and [13]).

In this way, the parametric equations according to formulas (3) and (4) can be written in the form

$$\begin{aligned} x(\tau) &= \xi(\tau)\cos\theta(\tau) - \eta(\tau)\sin\theta(\tau) \\ z(\tau) &= \xi(\tau)\sin\theta(\tau) + \eta(\tau)\cos\theta(\tau) \end{aligned} \tag{26}$$

in which the necessary ingredients are specified by the equations (21), (24) and (25). Actually, this is a new parametrization of the Cassinian ovals, which is different from those obtained previously (see, e.g., [2, 7]).

In case (II), taking into account equations (15), (6) and (8) – (10), we have

$$\kappa = \frac{\lambda}{r^3}, \qquad g = \frac{\lambda}{r}, \qquad \xi = r\dot{r}, \qquad \eta = \frac{\lambda}{r} + c \tag{27}$$

and

$$\dot{r}^2 = \frac{1}{r^4}P(r), \qquad P(r) = \left(r^2 + cr + \lambda\right)\left(r^2 - cr - \lambda\right). \tag{28}$$

In terms of a new variable τ such that

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{1}{r^2(\tau)} \tag{29}$$

as a consequence of relations (29), (27), (28), and (11) we have

$$\xi(\tau) = \frac{1}{r(\tau)}\frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau}, \qquad \eta(\tau) = \frac{\lambda}{r(\tau)} + c \tag{30}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = P(r) \tag{31}$$

and

$$\theta(\tau) = \int \frac{\lambda}{r(\tau)}\mathrm{d}\tau. \tag{32}$$

It should be noted that the polynomial $P(r)$ has four real roots $\alpha < \beta < \gamma < \delta$, which read

$$\begin{aligned} \alpha &= -\frac{1}{2}\left(\sqrt{c^2+4\lambda} - c\right), \qquad & \beta &= \frac{1}{2}\left(\sqrt{c^2+4\lambda} + c\right) \\ \gamma &= -\frac{1}{2}\left(\sqrt{c^2-4\lambda} + c\right), \qquad & \delta &= \frac{1}{2}\left(\sqrt{c^2-4\lambda} - c\right). \end{aligned}$$

Using the approach described in the Appendix one can write down the general form of a bounded solution to equation (31) as follows

$$r(\tau) = \beta + \frac{2\left[c^3 + \left(c^2+2\lambda\right)\sqrt{c^2+4\lambda} + 4c\lambda\right]\mathrm{sn}^2(\nu\tau,k)}{c^2 + \sqrt{c^4-16\lambda^2} - 2\left(c^2+2\lambda+c\sqrt{c^2+4\lambda}\right)\mathrm{sn}^2(\nu\tau,k)} \tag{33}$$

where

$$\nu = \frac{\sqrt{c^2+\sqrt{c^4-16\lambda^2}}}{2\sqrt{2}}, \qquad k = \sqrt{\frac{c^2-\sqrt{c^4-16\lambda^2}}{c^2+\sqrt{c^4-16\lambda^2}}}.$$

Solving the integral in the right hand side of equation (32) one obtains the explicit expression

$$\theta(\tau) = \frac{4\sqrt{2}\left[c^3+\left(c^2+2\lambda\right)\sqrt{c^2+4\lambda}+4c\lambda\right]\Pi_3(\tau)}{\left(\sqrt{c^2+4\lambda}+c\right)^2\sqrt{\sqrt{c^4-16\lambda^2}+c^2}} - \frac{\left[c\left(\sqrt{c^2+4\lambda}+c\right)+2\lambda\right]\tau}{\sqrt{c^2+4\lambda}+c} \tag{34}$$

for the slope angle, where

$$\Pi_3(\tau) = \Pi\left(-\frac{4\lambda}{c^2+\sqrt{c^4-16\lambda^2}}, \mathrm{am}(\nu\tau,k), k\right).$$

In this way, the parametric equations of the considered type of curves, which are of form (26), are obtained in an explicit form through the expressions (30), (33) and (34). Several examples of curves of curvature $\kappa = \lambda/r^3$, which can be thought of as trajectories of charged particles in the field of a magnetic dipole are drawn in Figure 2 using the foregoing parametric equations.

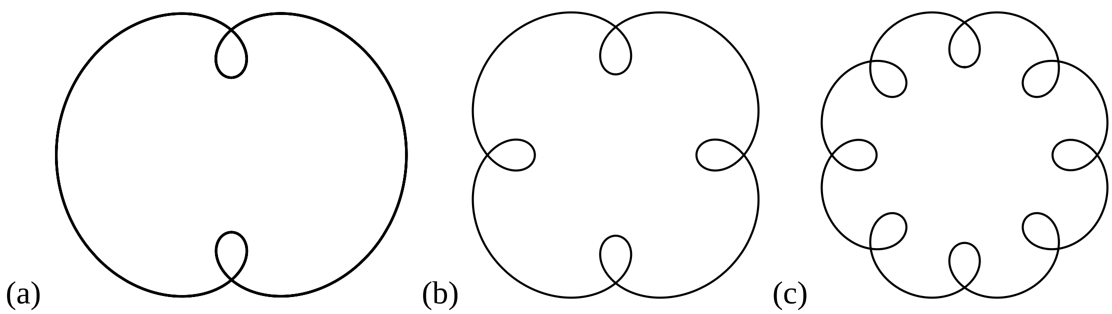

FIGURE 2. Closed plane curves of curvature (15) corresponding to $\mu = 0$ and $\lambda = 1$ drawn using the parametric equations (26), (30), (33), (34): (a) $c = -2.00449$; (b) $c = -2.04563$; (c) $c = -2.17243$.

APPENDIX

Consider the first-order nonlinear ordinary differential equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = a_0r^4+4a_1r^3+6a_2r^2+4a_3r+a_4 \tag{35}$$

where a_i ($i = 0,\ldots,4$) are real constants. The polynomial

$$P(r) = a_0r^4+4a_1r^3+6a_2r^2+4a_3r+a_4$$

appearing on the right hand side of equation (35) is of fourth degree with respect to the variable r, which allows, following [12, 13], to express the general solution of the foregoing nonlinear ordinary differential equation (provided that the polynomial $P(r)$ does not have multiple roots), in the form

$$r(\tau) = \rho + \frac{\sqrt{P(\rho)}\wp'(\tau, g_2, g_3) + \frac{1}{2}P'(\rho)\left(\wp(\tau, g_2, g_3) - \frac{1}{24}P''(\rho)\right) + \frac{1}{24}P(\rho)P''''(\rho)}{2\left(\wp(\tau, g_2, g_3) - \frac{1}{24}P''(\rho)\right)^2 - \frac{1}{48}P(\rho)P''''(\rho)} \tag{36}$$

where ρ is a constant, $\wp(\tau, g_2, g_3)$ is the Weierstrass elliptic function, τ is its argument and g_2 and g_3 are the so called invariants of the polynomial $P(r)$, which have the form

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4$$

and the primes indicate differentiations with respect to the arguments of the functions.

If $\mathring{r}$ is just a simple root of the polynomial $P(r)$, then the expression (36) simplifies significantly and takes the form

$$r(\tau) = \mathring{r} + \frac{1}{4}\frac{P'(\mathring{r})}{\wp(z, g_2, g_3) - \frac{1}{24}P''(\mathring{r})}. \tag{37}$$

The knowledge of the sign of the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{38}$$

of the Weierstrass elliptic function $\wp(\tau, g_2, g_3)$, the quartic $P(r)$ and the cubic polynomial

$$R(\tau) = 4\tau^3 - g_2\tau - g_3$$

is enough to express the solution of equation (35) in terms of Jacobi elliptic functions. First, let us remark that the polynomials $P(\phi)$ and $R(\tau)$ do not have multiple roots if and only if $\Delta \neq 0$, see [4]. In each such case the Weierstrass elliptic function appearing in the solution (37) to equation (35) can be expressed, cf. [1], as follows

(i) if $\Delta > 0$, then

$$\wp(\tau, g_2, g_3) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2\left(\sqrt{e_1 - e_3}\,\tau, m\right)}, \qquad m = \frac{e_2 - e_3}{e_1 - e_3} \tag{39}$$

where $e_1 > e_2 > e_3$ are the roots of the cubic polynomial $R(\tau)$, which, in this case, are real and $\mathrm{sn}(\cdot,\cdot)$ is the Jacobi sine function.

(ii) if $\Delta < 0$, then the polynomial $R(\tau)$ has one real root e_2 and a couple of complex conjugated roots e_1, e_3 and

$$\wp(\tau, g_2, g_3) = e_2 + H_2\frac{1 + \mathrm{cn}\left(2\tau\sqrt{H_2}, m\right)}{1 - \mathrm{cn}\left(2\tau\sqrt{H_2}, m\right)}, \quad m = \frac{1}{2} - \frac{3e_2}{4H_2}, \quad H_2 = \sqrt{3e_2^2 - \frac{g_2}{4}}. \tag{40}$$

where $\mathrm{cn}(\cdot,\cdot)$ is the Jacobi cosine function.

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