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Géométrie différentielle

ON THE GENERALIZED STURMIAN SPIRALS

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Abstract

The Sturm spirals which can be introduced as those plane curves whose curvature radius is equal to the distance from the origin are embedded into one-parameter family of curves. Explicit parametrization of the ordinary Sturmian spirals along with that of a wider family of curves are found and depicted graphically.

Key words: Sturm spirals, plane curves, equiangular spiral, logarithmic spiral, Norwich spiral

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1. Introduction. The fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [1], p. 296 or [8], p. 37). The simplicity of the situation, however, is elusive as in many cases it is impossible to find the curve explicitly. Having that in mind, it is clear that if the curvature is given as a function of its position, the situation is even more complicated. A nice exception is provided by the Euler's elastica curves [3, 6, 7] whose curvature actually is a function of the distance from a fixed line in the Euclidean plane. Viewing the Frenet-Serret equations as a fictitious dynamical system, it was proven in [11] that when the curvature is given just as a function of the distance from the origin the problem can always

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be reduced to quadratures. This last result is not entirely new as Singer [10] has already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and, therefore, the trajectories (the sought-after curves) can be found by the standard procedures in classical mechanics. However, the approach which we will follow here is entirely different from the group-theoretical [11] or mechanical [10] ones proposed in those papers. The method is illustrated upon the most natural example in the class of curves which curvatures are functions only of the distance from the origin. Here we consider the case in which the function in question is inversely proportional, namely,

$$\kappa = \frac{\sigma}{|\mathbf{x}|} = \frac{\sigma}{r} = \frac{\sigma}{\sqrt{x^2+z^2}}, \qquad \sigma > 0, \tag{1}$$

where x, z are the Cartesian coordinates in the plane XOZ which have to be considered as functions of the arc-length parameter s, and σ is assumed to be a positive real constant.

2. The Frenet-Serret equations. If $\theta(s)$ denotes the slope of the tangent to the curve with respect to the OX axis one has the following geometrical relations:

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta(s), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \sin\theta(s), \tag{2}$$

which can be deduced also from the Frenet-Serret equations (see also Fig. 1)

$$\frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} = \mathbf{T}(s), \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T}, \tag{3}$$

where $\mathbf{T}$ and $\mathbf{N}$ are the tangent and the normal vector to the curve respectively. Combining (1) and (2) we receive

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \frac{\sigma}{\sqrt{x^2+z^2}} \tag{4}$$

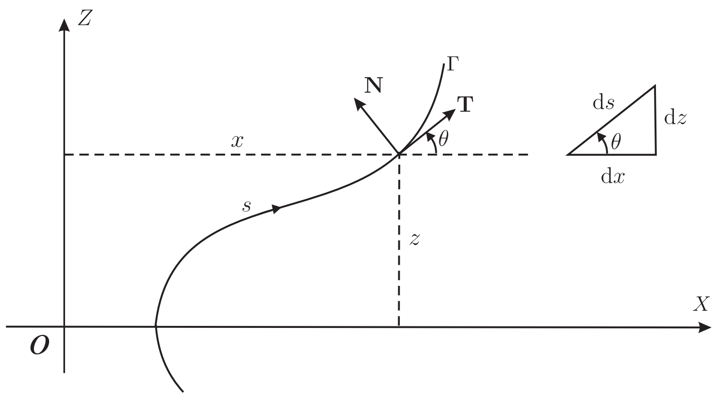

Fig. 1. Geometry of the plane curve

which is still a quite unpromising equation. We will proceed (as suggested but not pursued in [10]) by going to the co-moving frame associated with the curve

$$\mathbf{x} = \xi \mathbf{T} + \eta \mathbf{N} \tag{5}$$

and accordingly the Frenet-Serret equations (3) now read

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \dot{\xi} = \kappa\eta + 1, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = \dot{\eta} = -\kappa\xi, \qquad \kappa = \frac{\sigma}{\sqrt{\xi^2+\eta^2}}\cdot \tag{6}$$

3. Integration. Multiplying the first equation in (6) by ξ, the second one by η and summing the so obtained expressions, we find that

$$\xi = r\dot{r}, \tag{7}$$

where the dot means a differentiation with respect to the arc-length parameter. Substituting this expression back into equation (6) and integrating we obtain

$$\eta = -\int \kappa(r) r \mathrm{d}r + c, \tag{8}$$

where c is the integration constant. One should notice, however (cf. equation (5)), that the coordinates in the moving frame are not entirely independent but obey to the constraint

$$\xi^2 + \eta^2 = r^2, \tag{9}$$

which in view of equations (7) and (8) presents an ordinary differential equation for the radial coordinate.

4. Sturm spirals. By their very definition (cf. [13]) these plane curves possess the property that at each point their curvature radius $\mathcal{R}$ is equal to the distance r from the origin. Formulated in curvature terms this means that their curvature κ is given by formula (1) in which $\sigma \equiv 1$. Applying the scheme from the previous section one easily gets

$$\eta = -r + c \tag{10}$$

and

$$\dot{r} = \frac{\sqrt{2cr - c^2}}{r}, \qquad c > 0. \tag{11}$$

It is convenient to perform the integration of the above equation by switching to a new independent variable t defined by equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = r. \tag{12}$$

This leads to the following results:

$$r = \frac{c}{2}\left(t^2+1\right), \qquad \xi = ct, \qquad \eta = \frac{c}{2}\left(1-t^2\right). \tag{13}$$

Integration of the first equation in (2) gives us additionally that the new parameter t coincides (up to a real constant) with the slope angle, i.e.,

$$\theta = t. \tag{14}$$

By rewriting equation (5) in its components one has also the relations

$$x = \xi \cos\theta - \eta \sin\theta, \qquad z = \eta \cos\theta + \xi \sin\theta, \tag{15}$$

which combined with the above findings provides the sought parameterization of the Sturm spirals

$$x = c\left(t\cos t + \frac{t^2-1}{2}\sin t\right), \qquad z = c\left(\frac{1-t^2}{2}\cos t + t\sin t\right). \tag{16}$$

Making use of the above formulas one easily finds also the arc-length as a function of the parameter t, i.e.,

$$s = \frac{c}{2}\left(\frac{t^3}{3} + t\right). \tag{17}$$

By exchanging the numerical parameter c for 2ρ and taking into account the fundamental for this curve relation $\mathcal{R} \equiv r$, the above formula can be written into the form, which is nothing else but the intrinsic equation of the Sturm spiral

$$s = \frac{\mathcal{R} + 2\rho}{3}\sqrt{\frac{\mathcal{R} - \rho}{\rho}}. \tag{18}$$

5. Generalized Sturm spirals. Due to the restriction on the allowed values of σ one can consider as well the other two obvious possibilities, $\sigma > 1$ and $0 < \sigma < 1$ which have to be viewed as a generalization of the ordinary Sturmian spirals.

5.1. Case $\sigma > 1$. Here we will just outline the main ingredients of derivation (following again the scheme exposed in Section 3) starting with the equations

$$\eta = -\sigma r + c \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\sqrt{(1-\sigma^2)r^2 + 2c\sigma r - c^2}}{r}. \tag{19}$$

One easily concludes that the expression under the radical on the right-hand side is positive provided that $c > 0$ and r belongs either to finite or infinite interval, i.e.,

$$\frac{c}{\sigma+1} \le r \le \frac{c}{\sigma-1} \quad \text{and} \quad \sigma > 1 \qquad \text{or} \qquad r > \frac{c}{\sigma+1} \quad \text{and} \quad \sigma < 1. \tag{20}$$

As the subsection title suggests our immediate task is to consider the first of the possibilities presented above.

Exchanging as before (cf. equation (12)) the arc-length parameter with t leads to the formula

$$r = \frac{c}{\sigma^2-1}(\sigma + \sin\sqrt{\sigma^2-1}\,t), \qquad t \in \left[-\frac{\pi}{2\sqrt{\sigma^2-1}}, \frac{\pi}{2\sqrt{\sigma^2-1}}\right] \tag{21}$$

by which we find also

$$\xi = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{\sqrt{\sigma^2-1}}\cos\sqrt{\sigma^2-1}\,t, \qquad \theta = \sigma t. \tag{22}$$

Combining the above results with those from equation (21), the first equation in (19) and the general relations (15) give

$$\begin{aligned} x &= c\left(\frac{\cos\sqrt{\sigma^2-1}\,t\cos\sigma t}{\sqrt{\sigma^2-1}} + \frac{(\sigma\sin\sqrt{\sigma^2-1}\,t+1)\sin\sigma t}{\sigma^2-1}\right) \\ z &= c\left(\frac{\cos\sqrt{\sigma^2-1}\,t\sin\sigma t}{\sqrt{\sigma^2-1}} - \frac{\left(\sigma\sin\sqrt{\sigma^2-1}\,t+1\right)\cos\sigma t}{\sigma^2-1}\right). \end{aligned} \tag{23}$$

The expressions for the arc-length and the intrinsic equation in this case are

$$s = \frac{c}{\sigma^2-1}\left(\sigma t - \frac{\cos\sqrt{\sigma^2-1}\,t}{\sqrt{\sigma^2-1}} + \frac{\sigma\pi}{2\sqrt{\sigma^2-1}}\right) \tag{24}$$

and

$$\begin{aligned} s = \frac{c}{\sigma^2-1}\Bigg[&\frac{\sigma}{\sqrt{\sigma^2-1}}\arcsin\left[\frac{\sigma}{c}\left((\sigma^2-1)\mathcal{R}-c\right)\right] \\ &- \frac{1}{c}\sqrt{\sigma^2(1-\sigma^2)\mathcal{R}^2+2c\sigma^2\mathcal{R}-c^2} + \frac{\sigma\pi}{2\sqrt{\sigma^2-1}}\Bigg], \end{aligned} \tag{25}$$

where for the derivation of the last equation we have used the defining relation for the spiral which in this case states that $r = \sigma\mathcal{R}$.

A few remarks are in order here. First, while r takes its values in the interval (20) the variable t is running in the interval $\left[-\frac{\sigma\pi}{2\sqrt{\sigma^2-1}}, \frac{\sigma\pi}{2\sqrt{\sigma^2-1}}\right]$ and during this excursion the tangent to the curve turns to the angle $\frac{\sigma\pi}{\sqrt{\sigma^2-1}}$. This angle is greater than 2π for $\sigma < 2/\sqrt{3}$, equal to 2π for $\sigma = 2/\sqrt{3}$ and less than 2π for $\sigma > 2/\sqrt{3}$. All this is illustrated in Fig. 2.

5.2. Case $0 < \sigma < 1$. The first steps of the scheme given as in the previous case amounts to

$$\eta = -\sigma r + c \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\sqrt{(1-\sigma^2)r^2+2c\sigma r-c^2}}{r} \tag{26}$$

but one should keep in mind that now $r > \frac{c}{\sigma+1}$ and $\sigma < 1$. It turns out also more convenient to perform the integration of the equation on the right-hand side in (26) by introducing the parameter τ via the equation

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = r^2 \tag{27}$$

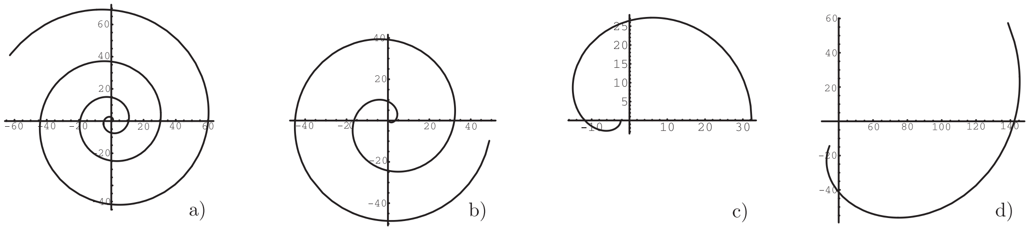

Fig. 2. The standard Sturmian spiral generated by (16) with $c = 0.25$ (a) and the generalized Sturmian spirals drawn via formulas in (23) with the following set of the parameters: (b) $c = 1$, $\sigma = 1.02$, (c) $c = 5$, $\sigma = 2/\sqrt{3}$ and (d) $c = 100$, $\sigma = 5/3$

and this produces

(28) $$r = \frac{c}{\sigma - \sin c\tau}, \qquad \tau \in \left[-\frac{\pi}{2c}, \frac{\arcsin\sigma}{c}\right]$$

and consequently

(29)
$$\xi = -\frac{c\cos c\tau}{\sigma - \sin c\tau}, \ \eta = -\frac{c\sin c\tau}{\sigma - \sin c\tau}, \ \theta(\tau) = \frac{\sigma}{\sqrt{1-\sigma^2}} \ln\frac{\sigma\tan\frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma\tan\frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1}.$$

Further, via equations (15) and (29) we obtain

(30)
$$x = \frac{c}{\sigma - \sin c\tau}\cos\left[c\tau - \frac{\sigma}{\sqrt{1-\sigma^2}}\ln\frac{\sigma\tan\frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma\tan\frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1}\right],$$
$$z = \frac{c}{\sigma - \sin c\tau}\sin\left[c\tau - \frac{\sigma}{\sqrt{1-\sigma^2}}\ln\frac{\sigma\tan\frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma\tan\frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1}\right],$$

and finally

(31) $$s = \frac{c\sigma}{(1-\sigma^2)^{3/2}} \ln\left[\frac{\sigma + (1+\sqrt{1-\sigma^2})\tan\left[\frac{1}{2}\arcsin\left(\frac{c}{r} - \sigma\right)\right]}{1 + \sqrt{1-\sigma^2} + \sigma\tan\left[\frac{1}{2}\arcsin\left(\frac{c}{r} - \sigma\right)\right]}\right] + \frac{\sqrt{(1-\sigma^2)r^2 + 2c\sigma r - c^2}}{1-\sigma^2}.$$

As before, one can easily obtain from the last expression the intrinsic equation of the curve by replacing r with $\sigma\mathcal{R}$.

5.3. Subcase $0 < \sigma < 1$ and $c = 0$. Just for completeness we will consider the situation when the integration constant c in previous subsection is zero. Obviously, the equations in (26) simplify to

(32) $$\eta = -\sigma r \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1-\sigma^2}.$$

The integration of the second one is immediate and gives

(33) $$r = \sqrt{1-\sigma^2}s + a$$

in which a denotes the integration constant that is necessarily positive. Applying the scheme it leads to the following results:

$$\xi = \left(1-\sigma^2\right)s + a\sqrt{1-\sigma^2}, \qquad \eta = -\sigma\left(\sqrt{1-\sigma^2}s + a\right) \tag{34}$$

and

$$\theta(s) = \frac{\sigma}{\sqrt{1-\sigma^2}}\ln(\sqrt{1-\sigma^2}s + a) \tag{35}$$

which allow to write down the explicit parametrization of the corresponding spiral

$$\begin{aligned} x &= ((1-\sigma^2)s + a\sqrt{1-\sigma^2})\cos\theta(s) + \sigma(\sqrt{1-\sigma^2}s + a)\sin\theta(s), \\ z &= ((1-\sigma^2)s + a\sqrt{1-\sigma^2})\sin\theta(s) - \sigma(\sqrt{1-\sigma^2}s + a)\cos\theta(s). \end{aligned} \tag{36}$$

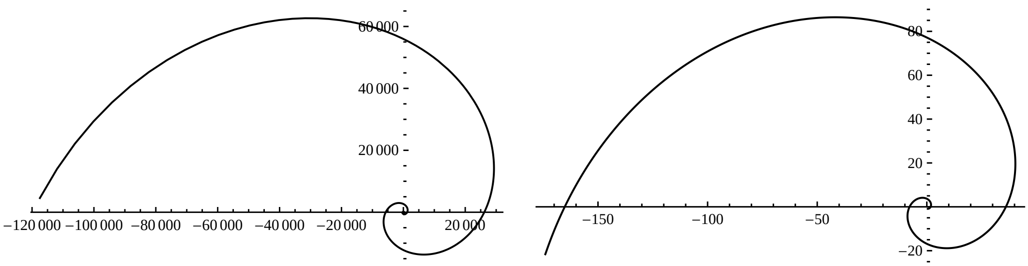

Fig. 3. Sturmian spirals produced via formulas (30) with $\sigma = 0.9$, $c = 1$ (left) and formulas (36) with $\sigma = 0.9$, $a = 1$ (right)

6. Concluding remarks. Among the principal motivations for writing this paper we should mention the question by Professor S. Woronowicz (Warsaw University) who, during the conference talk of the first named author at Bialowieza Conference (Poland), asked which is the curve whose curvature is σ/r? The answer (which at that time was actually a guess) was that probably these curves are spirals. It was a kind of surprise to learn that they do not coincide with any of the famous spirals like those bearing the names of Archimedes, Cotes, Euler (Cornu), Fermat, Galileo, Nielsens, Poinsot, etc. that are traditionally treated in the textbooks on classical differential geometry [1,4,5,8,9]. Only in the book by Zwikker [13] we have found a short note about Norwich or Sturm spiral discussed here in Section 4. More thorough search, however, clarifies that the equiangular spiral

$$x = me^{\phi\cot\alpha}\cos\phi, \qquad z = me^{\phi\cot\alpha}\sin\phi, \qquad m, \alpha \in \mathbb{R}^+, \quad \phi \in \mathbb{R} \tag{37}$$

discovered by Descartes and sometimes also called logarithmic spiral possesses a curvature given by the formula $\kappa = \sin\alpha/r$ and belongs to the class studied in the last two sections. The name of the curve comes from its property to cut radius vectors from the origin at a constant angle α. This seems to be also the reason excepting the fact that the insects approach the candle along this curve, thinking perhaps they are flying in a straight line at a constant angle to the rays of the light [2]. Among other interesting properties of this curve we will mention that successive generation of its evolutes, pedal curves or inverses are still equiangular spirals (for more details see [12]).

It was a challenging task to provide here a detailed description from the first principles of the family of Sturm spirals having so nice geometrical characteristic and to supply their explicit parameterizations.

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