VECTOR DECOMPOSITION OF FINITE ROTATIONS

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On the basis of intrinsic properties of the vector parameterization of rotational motions this work presents an explicit solution of the problem of decomposition of any finite rotation into a product of three successive finite rotations about prescribed axes.

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1. Introduction

The presentation of rigid body displacements is an old mechanical problem which is of great importance for solving different real tasks. In geometrical and computational mechanics the rigid body kinematics is considered in terms of vectors and matrices. In this aspect the rotation group in a three-dimensional real space may be described by combining our knowledge from analytical mechanics, vector analysis, algebra and differential geometry. Here is the place to mention that the different parameterizations of the rotation group $\mathcal{SO}(3)$ influence significantly the efficiency of the kinematical and dynamical models of the rigid body and multibody mechanical systems as well. Various analytical representations of rotations are obtained when the rotation is expressed by defining its action on vectors, matrices, quaternions or spinors [3]. This problem is considered in detail in the review papers [2, 15] and [16]. Most frequently the parameterizations of rotations are expressed via Eulerian angles as in the classical 3-1-3 case and all other combinations like 3-2-3, 2-1-2, etc., and in the case of nonrepeating axes by the Bryant–Cardan–Krylov angles. Alternatively, one can use the Euler–Rodrigues or Cayley–Klein parameters. All of them have found a lot of applications in various scientific and technological areas.

The numbers 1, 2, 3 appearing in the triples listed above mean rotations about the axis OX, OY and OZ, respectively. For instance, the new frame $\tilde{\mathcal{F}}$ is related to the old one $\mathcal{F}$ (each of them is specified by three orthonormal vectors) through the direction cosine matrix $\mathcal{R}$,

$$\tilde{\mathcal{F}} = \mathcal{R}(\psi, \theta, \phi)\mathcal{F}, \tag{1}$$

where ψ, θ and ϕ are the rotation angles. For definiteness, in the 3-2-1 case the matrix $\mathcal{R} = \mathcal{R}(\psi, \theta, \phi)$ is just the matrix product

$$\mathcal{R}(\psi, \theta, \phi) = \mathcal{R}_X(\phi)\mathcal{R}_Y(\theta)\mathcal{R}_Z(\psi). \tag{2}$$

To find the resultant axis and the angle of rotation after two, three or more finite successive rotations is an important and almost trivial problem in multibody mechanics. The inverse problem however, namely to decompose a finite rotation into three consecutive rotations about prescribed axes is a more difficult but quite important problem for motion planning in the group of rotations and the inverse kinematic problem for manipulator systems. The present paper gives explicit formulae for this problem using a vector-like parameterization of the rotation group. Its organization is as follows.

In the first part the notion of vector-parameter is introduced and its properties are given. After that the above-mentioned problem is solved and exemplified in concrete settings. Actually, the method is realized as an analytical algorithm using the computer algebra system `Mathematica` for symbolic calculations.

2. Vector representation of rotation motions

Let us consider the special orthogonal Lie group $\mathcal{SO}(3)$ presenting the rotational motions

$$\mathcal{SO}(3) = \{\mathcal{R} \in \mathsf{GL}(3, \mathbb{R}) \ ; \ \mathcal{R}\mathcal{R}^T = I, \ \det\mathcal{R} = 1\} \tag{3}$$

where $\mathsf{GL}(3, \mathbb{R})$ denotes the space of nonsingular 3×3 real matrices and "T" is the symbol for transposition of a matrix. The Lie algebra $\mathfrak{so}(3)$ of $\mathcal{SO}(3)$ (i.e. its infinitesimal generators) consists of all real skew-symmetric 3×3 matrices. If a matrix A belongs to the Lie algebra $\mathfrak{so}(3)$, the matrix $I - A$ is invertible, and the Cayley transformation making the connection between the algebra and the group explicit is given by the formulae [19]

$$\mathcal{R} = (I + A)(I - A)^{-1} = (2I - (I - A))(I - A)^{-1} = 2(I - A)^{-1} - I.$$

In the particular case of three-dimensional space, there exists a map between vectors and the space $\mathcal{A}(3)$ of the three-dimensional skew-symmetric matrices, i.e. if $\mathbf{c} \in \mathbb{R}^3$, we have

$$\mathbf{c} \quad \longleftrightarrow \quad \mathbf{c}^\times = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}, \qquad \mathbf{c}^\times \in \mathcal{A}(3). \tag{4}$$

This isomorphism of vector spaces can be extended to isomorphism of Lie algebras of vectors in $\mathbb{R}^3$ equipped with the standard cross product of vectors (denoted by $\times$) as a Lie algebra operation, and the algebra of antisymmetric matrices $\mathcal{A}(3)$ in which the Lie bracket is just the commutator, i.e.

$$(\mathbb{R}^3, \times) \longleftrightarrow (\mathcal{A}(3), [\cdot\,, \cdot]). \tag{5}$$

This can be checked by performing the following computation:

$$\mathbf{a} \times \mathbf{c} \longleftrightarrow (\mathbf{a} \times \mathbf{c})^{\times} = [\mathbf{a}^{\times}, \mathbf{c}^{\times}] = \mathbf{a}^{\times}\mathbf{c}^{\times} - \mathbf{c}^{\times}\mathbf{a}^{\times}, \qquad \mathbf{a}, \mathbf{c} \in \mathbb{R}^3, \qquad \mathbf{a}^{\times}, \mathbf{c}^{\times} \in \mathcal{A}(3).$$

Linearity is obvious and that the Jacobi identities are satisfied in both spaces is also a matter of straightforward calculations.

All this allows to write down the $\mathcal{SO}(3)$ matrix in the form

$$\mathcal{R} = \mathcal{R}(\mathbf{c}) = \mathcal{R}_{\mathbf{c}} = (I + \mathbf{c}^{\times})(I - \mathbf{c}^{\times})^{-1} = \frac{(1 - \mathbf{c}^2)I + 2\,\mathbf{c} \otimes \mathbf{c} + 2\,\mathbf{c}^{\times}}{1 + \mathbf{c}^2} \tag{6}$$

and to consider it as a mapping from $\mathbb{R}^3$ to $\mathcal{SO}(3)$ for which the smooth inverse is given by the formula

$$\mathbf{c}^{\times} = \frac{\mathcal{R} - \mathcal{R}^T}{1 + \operatorname{tr}(\mathcal{R})}\,\cdot \tag{7}$$

Here I is a 3×3 identity matrix, $\mathbf{c}^2 = \mathbf{c}.\mathbf{c}$ and $\mathbf{c} \otimes \mathbf{c}$ denote the dot, respectively the diadic product of the vector $\mathbf{c}$ with itself, and $\operatorname{tr}(\mathcal{R})$ is the trace of $\mathcal{R}$. The formula above provides us with an explicit parameterization of $\mathcal{SO}(3)$. The vector $\mathbf{c}$ is called the *vector-parameter*. It is parallel to the axis of rotation and its modulus $\|\mathbf{c}\|$ is equal to $\tan(\alpha/2)$ where α is the angle of rotation about the same axis. The so defined vector-parameters form a Lie group with the following composition law

$$\mathbf{c} = \langle \mathbf{c}_1, \mathbf{c}_2 \rangle = \frac{\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_1 \times \mathbf{c}_2}{1 - \mathbf{c}_1 . \mathbf{c}_2}\,\cdot \tag{8}$$

Every component of $\mathbf{c}$ can take all real values from $-\infty$ to $+\infty$ without any restrictions, which is a great advantage compared with the obvious asymmetry in the Eulerian parameterization (see e.g. [8, 10, 18]). The vector $\mathbf{c} \equiv 0$ corresponds to the identity matrix $\mathcal{R}(0) \equiv I$ and the opposite vector $-\mathbf{c}$ produces the inverse rotation $\mathcal{R}(-\mathbf{c}) \equiv \mathcal{R}^{-1}(\mathbf{c})$. The operation of conjugation in $\mathcal{SO}(3)$ leads to linear transformations in the vector-parameter space

$$\mathcal{R}(\mathbf{c})\,\mathcal{R}(\mathbf{c}')\,\mathcal{R}^{-1}(\mathbf{c}) = \mathcal{R}(\mathbf{c}'')$$

where $\mathbf{c}'' = \mathcal{R}(\mathbf{c})\,\mathbf{c}' = \mathcal{R}_{\mathbf{c}}\,\mathbf{c}'$. Such a parameterization in the theory of Lie groups is called a natural one. It is worth mentioning also that no other parameterization possesses either this property or a manageable superposition law. The exceptional case in (8), i.e. when $\mathbf{c}_1 . \mathbf{c}_2 = 1$ means that the resultant vector $\mathbf{c}$ produces a rotation at 180° and may be treated by replacing $\mathbf{c}$ with a vector $\mathbf{d}$ from another chart of the $\mathcal{SO}(3)$ group manifold using the relation

$$\mathbf{d} = \frac{\mathbf{c}}{1+\mathbf{c}^2} \tag{9}$$

and

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}'(\mathbf{d}) = I + 2(\sqrt{1-\mathbf{d}^2} + \mathbf{d}^{\times})\mathbf{d}^{\times}. \tag{10}$$

The direction of $\mathbf{d}$ coincides with that of $\mathbf{c}$ and $\|\mathbf{d}\| = \sin(\alpha/2)$. The composition law in this exceptional case is

$$\mathbf{d} = \langle \mathbf{d}_1\,,\ \mathbf{d}_2 \rangle = \sqrt{1-\mathbf{d}_2^2}\,\mathbf{d}_1 + \sqrt{1-\mathbf{d}_1^2}\,\mathbf{d}_2 + \mathbf{d}_1 \times \mathbf{d}_2. \tag{11}$$

The vector-parameter is known also as the Gibbs or Rodrigues vector. Some authors call it also a vector of finite rotations. Nice discussions and historical facts on the problem of orientation mapping are given in [7]. Vector representation of rotations in three-dimensional space $\mathbb{R}^3$ is a principal subject of consideration in many papers and by no means exhaustive list includes [1, 2, 4, 9, 20], and references therein. It should be mentioned however that after Fedorov [6] who uses the vector-parameterization in problems related to the Lorentz group, the present authors are among the first who make use of it in the classical [12] and quantum mechanics [11]. Later on the vector-parameter was applied also in modeling and control of open-loop mechanical systems like manipulators [13, 14], vehicle devices and biomechanical systems [15].

The most important properties of the vector-parameterization of the rotation group and its composition law are presented in a nutshell below:

$$\begin{aligned}
\langle \mathbf{c}, \mathbf{c}' \rangle &= \mathbf{c}'' = \frac{\mathbf{c} + \mathbf{c}' + \mathbf{c} \times \mathbf{c}'}{1 - \mathbf{c}.\mathbf{c}'},\\
\langle \mathbf{c}, \mathbf{0} \rangle &= \langle \mathbf{0}, \mathbf{c} \rangle = \mathbf{c},\\
\langle \mathbf{c}, -\mathbf{c} \rangle &= 0,\\
\langle \mathbf{c}', \mathbf{c} \rangle &\neq \langle \mathbf{c}, \mathbf{c}' \rangle, \qquad \mathbf{c}' \neq \lambda \mathbf{c}, \quad \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\},\\
\langle \langle \mathbf{a},\ \mathbf{b} \rangle,\ \mathbf{c} \rangle &= \langle \mathbf{a},\ \langle \mathbf{b},\ \mathbf{c} \rangle \rangle = \langle \mathbf{a},\ \mathbf{b},\ \mathbf{c} \rangle,\\
-\langle \mathbf{a},\ \mathbf{b} \rangle &= \langle -\mathbf{b},\ -\mathbf{a} \rangle.
\end{aligned} \tag{12}$$

From the alternative expression of the Cayley formula

$$\mathcal{R} = (I - A)(I + A)^{-1} \tag{13}$$

one can obtain the composition law in the case when the first rotation is generated by $\mathbf{c}_1$, i.e.

$$\tilde{\mathbf{c}} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_2.\mathbf{c}_1} = \frac{\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_1 \times \mathbf{c}_2}{1 - \mathbf{c}_1.\mathbf{c}_2}. \tag{14}$$

Generally speaking, we have

$$\mathbf{c}_+ = \frac{\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_1 \times \mathbf{c}_2}{1 - \mathbf{c}_1.\mathbf{c}_2}, \qquad \mathbf{c}_- = \frac{\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_1 \times \mathbf{c}_2}{1 - \mathbf{c}_1.\mathbf{c}_2}. \tag{15}$$

Both vector-parameters $\mathbf{c}_+$ and $\mathbf{c}_-$ correspond to the composition of two vectors in the reverse order, and they are situated symmetrically with respect to the plane defined by $\mathbf{c}_1$ and $\mathbf{c}_2$.

3. Statement of the problem

For a given vector-parameter $\mathbf{c}$ (i.e. its direction and rotation angle are known), and arbitrarily chosen directions specified via three unit vectors $\hat{\mathbf{c}}_1$, $\hat{\mathbf{c}}_2$ and $\hat{\mathbf{c}}_3$, the problem is to find the respective partial rotation angles about them which when performed consecutively the rotation corresponding to $\mathbf{c}$.

This problem is very important for many real tasks like: manipulator control, spacecraft dynamics and control, rehabilitation procedures, etc. In pure technical aspects the problem is connected with the choice of appropriate motors and safety measures.

4. Solution

Introducing

$$\mathbf{c}_1 = u\,\hat{\mathbf{c}}_1, \qquad \mathbf{c}_2 = v\,\hat{\mathbf{c}}_2, \qquad \mathbf{c}_3 = w\,\hat{\mathbf{c}}_3 \tag{16}$$

where $\hat{\mathbf{c}}_i$, $i = 1, 2, 3$, are the unit vectors along the prescribed axes of rotations makes obvious that the problem is reduced to finding the parameters u, v and w because according to the very definition we have

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_3). \tag{17}$$

Now, the appropriate multiplications of Eq. (17) give us the following relations,

$$(\hat{\mathbf{c}}_1,\ \mathcal{R}(\mathbf{c})\hat{\mathbf{c}}_3) = (\hat{\mathbf{c}}_1, \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3) = (\hat{\mathbf{c}}_1,\ \mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3), \tag{18}$$

$$(\hat{\mathbf{c}}_2,\ \mathcal{R}(\mathbf{c})\hat{\mathbf{c}}_3) = (\hat{\mathbf{c}}_2,\ \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3) = (\mathcal{R}^T(\mathbf{c}_1)\hat{\mathbf{c}}_2, \mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3), \tag{19}$$

$$(\hat{\mathbf{c}}_1,\ \mathcal{R}(\mathbf{c})\hat{\mathbf{c}}_2) = (\hat{\mathbf{c}}_1,\ \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_3)\hat{\mathbf{c}}_2) = (\mathcal{R}^T(\mathbf{c}_2)\hat{\mathbf{c}}_1,\ \mathcal{R}(\mathbf{c}_3)\hat{\mathbf{c}}_2), \tag{20}$$

$$(\hat{\mathbf{c}}_3,\ \mathcal{R}(\mathbf{c})\hat{\mathbf{c}}_3) = (\hat{\mathbf{c}}_3,\ \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3) = (\mathcal{R}^T(\mathbf{c}_1)\hat{\mathbf{c}}_3,\ \mathcal{R}(\mathbf{c}_2)\hat{\mathbf{c}}_3), \tag{21}$$

where $(\cdot\ ,\ \cdot)$ denotes the dot product of vectors. For obtaining the above equations, besides of (12), we used another very useful property of the vector-parameters, namely

$$\mathcal{R}(\mathbf{c})\,\mathbf{c} = \mathcal{R}^T(\mathbf{c})\,\mathbf{c} = \mathbf{c}, \tag{22}$$

which is valid for any vector-parameter $\mathbf{c}$. In this setting the solution of the problem is almost immediate. First, one can find $v = v(\mathbf{c}, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$ from Eq. (18). After that, using either Eq. (19) or Eq. (21) one can determine $u = u(v, \mathbf{c}, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$, and finally Eq. (20) gives $w = w(v, \mathbf{c}, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3)$. It is seen that the parameters u and w are independent of each other. Actually, any of the unknown quantities u, v and

w can be obtained by solving the respective quadratic equation given below,

$$A_1\, u^2 + B_1\, u + C_1 = 0, \tag{23}$$

$$A_2\, v^2 + B_2\, v + C_2 = 0, \tag{24}$$

$$A_3 w^2 + B_3 w + C_3 = 0. \tag{25}$$

Notice however that in order to find the parameters u and w one has to solve in advance equation (24) for v.

5. Basic notation and algorithm

The proposed algorithm is realized by using of the following notation

$$\begin{aligned}
&C_{ij} := (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j) \;=\; (\hat{\mathbf{c}}_j, \hat{\mathbf{c}}_i) \;=\; C_{ji}, \qquad i \neq j := 1,\ 2,\ 3,\\
&C_{00} := (\mathbf{c}, \mathbf{c}), \qquad C_{0i} = (\mathbf{c}, \hat{\mathbf{c}}_i) = (\hat{\mathbf{c}}_i, \mathbf{c}) = C_{i0},\\
&V_{ijk} := (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j \times \hat{\mathbf{c}}_k),\\
&V_{123} := (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3) = V_{312} = -V_{321} = V,\\
&V_{i0j} := (\hat{\mathbf{c}}_i, \mathbf{c} \times \hat{\mathbf{c}}_j) = F_{ij},\\
&H := (\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3),
\end{aligned} \tag{26}$$

where $\times$ means the cross product. The coefficients of the quadratic equations are respectively

$$\begin{aligned}
A_1 &= \big(C_{23} + C_{02}\, C_{03} + (1 + C_{00})\, (C_{12}\, C_{13} - 2\, C_{12}^2\, C_{23}) + F_{23}\big)\, v^2\\
&\quad -2\, (1 + C_{00})\, C_{12}\, V\, v + C_{23} + C_{02}\, C_{03} - C_{12}\, C_{13} - C_{00}\, C_{12}\, C_{13} + F_{23},\\
B_1 &= -\, (1 + C_{00})\, (V\, v^2 - 2\, H\, v - V),\\
C_1 &= (C_{02}\, C_{03} - C_{00}\, C_{23} + F_{23})\, (v^2 + 1),
\end{aligned} \tag{27}$$

$$\begin{aligned}
A_2 &= C_{13} + C_{01}\, C_{03} - C_{12}\, C_{23} - C_{00}\, C_{12}\, C_{23} + F_{13},\\
B_2 &= -\, (1 + C_{00})\, V,\\
C_2 &= C_{01}\, C_{03} - C_{00}\, C_{13} + F_{13},
\end{aligned} \tag{28}$$

$$\begin{aligned}
A_3 &= \big(C_{12} + C_{01}\, C_{02} + (1 + C_{00})\, (C_{13}\, C_{23} - 2\, C_{12}\, C_{23}^2) + F_{12}\big)\, v^2\\
&\quad -2\, (1 + C_{00})\, C_{23}\, V\, v + C_{12} + C_{01}\, C_{02} - C_{13}\, C_{23} - C_{00}\, C_{13}\, C_{23} + F_{12},\\
B_3 &= -\, (1 + C_{00})\, (V\, v^2 - 2\, H\, v - V),\\
C_3 &= (C_{01}\, C_{02} - C_{00}\, C_{12} + F_{12})\, (v^2 + 1).
\end{aligned} \tag{29}$$

So the solutions of the quadratic equations (23) and (25)

$$v_1 \longrightarrow \begin{cases} u_1(v_1),\ u_2(v_1),\\ w_1(v_1),\ w_2(v_1), \end{cases} \qquad \text{and} \qquad v_2 \longrightarrow \begin{cases} u_1(v_2),\ u_2(v_2),\\ w_1(v_2),\ w_2(v_2), \end{cases}$$

depend on the solutions of Eq. (24) given by the formula

$$v_{1,2} = \frac{-B_2 \mp \sqrt{B_2^2 - 4\,A_2\,C_2}}{2\,A_2} \tag{30}$$

and in the same manner

$$\begin{aligned} u_{1,2}\,(v_1) &= \frac{-B_1 \mp \sqrt{B_1^2 - 4\,A_1\,C_1}}{2\,A_1}\,\Big|_{v=v_1}, \\ w_{1,2}\,(v_1) &= \frac{-B_3 \mp \sqrt{B_3^2 - 4\,A_3\,C_3}}{2\,A_3}\,\Big|_{v=v_1}, \\ u_{1,2}\,(v_2) &= \frac{-B_1 \mp \sqrt{B_1^2 - 4\,A_1\,C_1}}{2\,A_1}\,\Big|_{v=v_2}, \\ w_{1,2}\,(v_2) &= \frac{-B_3 \mp \sqrt{B_3^2 - 4\,A_3\,C_3}}{2\,A_3}\,\Big|_{v=v_2}\,. \end{aligned} \tag{31}$$

Let us remark that we will have real solution(s) of Eq. (30) provided that the condition

$$B_2^2 - 4A_2C_2 \geq 0 \tag{32}$$

is fulfilled and this means that inequality (32) is the necessary condition for the start of the algorithm. In the most favorable case we will have eight solutions (i.e. vectors) which are given below:

$$\begin{array}{ll} \langle u_1(v_1)\hat{\mathbf{c}}_1,\ v_1\hat{\mathbf{c}}_2,\ w_1(v_1)\hat{\mathbf{c}}_3\rangle, & \langle u_1(v_2)\hat{\mathbf{c}}_1,\ v_2\hat{\mathbf{c}}_2,\ w_1(v_2)\hat{\mathbf{c}}_3\rangle, \\ \langle u_2(v_1)\hat{\mathbf{c}}_1,\ v_1\hat{\mathbf{c}}_2,\ w_1(v_1)\hat{\mathbf{c}}_3\rangle, & \langle u_2(v_2)\hat{\mathbf{c}}_1,\ v_2\hat{\mathbf{c}}_2,\ w_1(v_2)\hat{\mathbf{c}}_3\rangle, \\ \langle u_2(v_1)\hat{\mathbf{c}}_1,\ v_1\hat{\mathbf{c}}_2,\ w_2(v_1)\hat{\mathbf{c}}_3\rangle, & \langle u_2(v_2)\hat{\mathbf{c}}_1,\ v_2\hat{\mathbf{c}}_2,\ w_2(v_2)\hat{\mathbf{c}}_3\rangle, \\ \langle u_1(v_1)\hat{\mathbf{c}}_1,\ v_1\hat{\mathbf{c}}_2,\ w_2(v_1)\hat{\mathbf{c}}_3\rangle, & \langle u_1(v_2)\hat{\mathbf{c}}_1,\ v_2\hat{\mathbf{c}}_2,\ w_2(v_2)\hat{\mathbf{c}}_3\rangle. \end{array}$$

It turns out however that only two of the vectors in the above list are actually solutions of the problem, i.e.

$$\langle u\,\hat{\mathbf{c}}_1,\ v\,\hat{\mathbf{c}}_2,\ w\,\hat{\mathbf{c}}_3\rangle\ \equiv\ \mathbf{c}. \tag{33}$$

The criteria by which one of them has to be selected is dictated by the energy (time) consumption needed for the physical realization of the motion. It is clear that neither energy nor time depend on the sign of the rotation angle and therefore it should be an even function of the parameters u, v and w. As a penalty (cost) function we propose

$$\mathcal{C}(u, v, w) = u^2 + v^2 + w^2 \tag{34}$$

as it meets the above requirement. Let us remark also that after solving the general decomposition problem, we may consider the obtained parameters u, v, w as the generalized Euler angles [5, 17].

6. Examples

6.1. Numerical tests

According to the authors knowledge the only published examples of numerical decomposition of rotational matrices can be traced back to the paper by Wohlhart [20] and it should be mentioned that the results there rely on an entirely different algorithm. Translated into our notation the initial data used in [20] are as follows:

$$\begin{aligned} \mathbf{c} &:= \tan 30^0 (\cos 50^0 \cos 25^0, \cos 50^0 \sin 25^0, \sin 50^0), \\ \hat{\mathbf{c}}_1 &:= 80^0 \cos 45^0, \cos 80^0 \sin 45^0, \sin 80^0), \\ \hat{\mathbf{c}}_2 &:= (\sin 60^0, \cos 60^0, 0), \\ \hat{\mathbf{c}}_3 &:= (1, 0, 0). \end{aligned} \tag{35}$$

The pair of solutions in this case is

$$\begin{aligned} u_1 &= -0.106955, & v_1 &= 157.192, & w_1 &= -2.73183, \\ u_2 &= 0.45189, & v_2 &= -0.0392637, & w_2 &= 0.303141. \end{aligned} \tag{36}$$

Converted into radians these solutions reproduce the results presented in [20]. The values of the penalty function (34) for the set of the above solutions are 24716.8 and 0.297641 respectively making clear that the second solution would be much more easily effectuated by motors.

Wolhart [20] treated also the case when the same rotation is resolved into the form of three successive rotations that are carried about the axes $\hat{\mathbf{c}}_3$, $\hat{\mathbf{c}}_2$ and again about $\hat{\mathbf{c}}_3$, i.e.

$$\begin{aligned} \mathbf{c} &:= \tan 30^0 (\cos 50^0 \cos 25^0, \cos 50^0 \sin 25^0, \sin 50^0), \\ \hat{\mathbf{c}}_1 &:= (1, 0, 0), \\ \hat{\mathbf{c}}_2 &:= (\sin 60^0, \cos 60^0, 0), \\ \hat{\mathbf{c}}_3 &:= (1, 0, 0). \end{aligned} \tag{37}$$

This time the pair of solutions is

$$\begin{aligned} u_1 &= -0.369392, & v_1 &= -1.39519, & w_1 &= 76.5567, \\ u_2 &= 0.350947, & v_2 &= 1.39519, & w_2 &= -1.24092 \end{aligned} \tag{38}$$

and the penalty function (cost time) for them amounts respectively to 5863.01 and 3.60958.

6.2. Cardan angles

The proposed algorithm simplifies significantly when the prescribed axes are specified by a system of three mutually orthogonal vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ (this case refers to a right-handed orthonormal triad). In these circumstances (known also as the Cardan or Bryant parameterizations) we have $C_{ij} = 0$, $V = 1$, $H = 0$. Then

$$\begin{aligned} A_1 &= (C_{02}\, C_{03} + F_{23})\,(v^2 + 1), \\ B_1 &= -\,(1 + C_{00})\,(v^2 - 1), \\ C_1 &= (C_{02}\, C_{03} + F_{23})\,(v^2 + 1), \end{aligned} \tag{39}$$

$$\begin{aligned} A_2 &= C_{01}\, C_{03} + F_{13}, \\ B_2 &= -\,(1 + C_{00}), \\ C_2 &= C_{01}\, C_{03} + F_{13}, \end{aligned} \tag{40}$$

$$\begin{aligned} A_3 &= (C_{01}\, C_{02} + F_{12})\,(v^2 + 1), \\ B_3 &= -\,(1 + C_{00})\,(v^2 - 1), \\ C_3 &= (C_{01}\, C_{02} + F_{12})\,(v^2 + 1). \end{aligned} \tag{41}$$

Any of the six sets of Euler angles sets which have nonrepeated indices (i.e. 1-2-3, 3-2-1, 2-3-1, 3-1-2, 2-1-3, 1-3-2) is covered by this specialization. In particular, the 3-2-1 case could be found in the literature under different names like Cardan angles, Bryant angles, or "yaw, pitch, roll" set. This kind of parameterization is quite suitable for manipulator end-effector description, or in aircraft and spacecraft applications.

6.3. Euler angles

This case was already considered numerically with concrete data but it deserves more abstract treatment as well. More concretely, here we have in mind the decomposition of the rotation into three consecutive rotations about axes which are mutually orthogonal but this time with repeated indices (i.e. 1-2-1, 1-3-1, 2-1-2, 2-3-2, 3-1-3, 3-2-3). The most popular choice in mechanics is the 3-1-3 sequence (the standard Euler angles) for which we have the following obvious reductions: $C_{13} = 1$, $C_{12} = C_{23} = V = F_{13} = 0$ and $H = -1$.

Then, the corresponding coefficients A_i, B_i, C_i $(i = 1, 2, 3)$ in the formulae (27), (28) and (29) are reduced accordingly to

$$\begin{aligned} A_1 &= (C_{02}\, C_{03} - F_{12})\,(v^2 + 1), \\ B_1 &= 2\,(1 + C_{00})\,v, \\ C_1 &= (C_{02}\, C_{03} - F_{12})\,(v^2 + 1), \end{aligned} \tag{42}$$

$$\begin{aligned} A_2 &= C_{01}^2 + 1, \\ B_2 &= 0, \\ C_2 &= C_{01}^2 - C_{00}, \end{aligned} \tag{43}$$

$$\begin{aligned} A_3 &= (C_{01}^2 + F_{12})\,(v^2 + 1), \\ B_3 &= 2\,(1 + C_{00})\,v, \\ C_3 &= (C_{01}^2 + F_{12})\,(v^2 + 1). \end{aligned} \tag{44}$$

A straightforward computation in this setting shows that the discriminant in (30) amounts to

$$4(C_{01}^2 + 1)(C_{00} - C_{01}^2) = 4((\mathbf{c}.\hat{\mathbf{c}}_1)^2 + 1)\left\|\mathbf{c} \times \hat{\mathbf{c}}_1\right\|^2 \tag{45}$$

which is definitely positive and this ensures that any two orthogonal axes are enough for Euler type decomposition of an arbitrary rotational matrix. This has already been discussed in [4, 5, 17] but the proof here is based upon more elementary techniques.

7. Conclusion

A method for analytical and numerical decomposition of finite rotations about three vectors along prescribed axes is presented. It is valid in the cases when the three axes of rotation are given in general form by their unit vectors. Two special numerical cases are considered in some detail and two general cases associated with Cardan and Euler angles are discussed as well. The proposed algorithm is a definite procedure which is quite convenient for kinematical investigations in the area of spacecraft dynamics, theory of mechanisms, robotics, biomechanics, etc. In some sense the results of the paper confirm the usefulness of the idea of introducing generalized Euler angles associated with three arbitrarily chosen axes because it could be considered as a method for introducing such charts on the manifold of the rotation group. The numerical and analytical algorithms are realized as `Mathematica` routines.

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