Traveling Wave Solutions of the Gardner Equation and Motion of Plane Curves Governed by the mKdV Flow

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Abstract. The Gardner equation is well-known in the mathematical literature since the late sixties of 20th century. Initially, it appeared in the context of the construction of local conservation laws admitted by the KdV equation. Later on, the Gardner equation was generalized and found to be applicable in various branches of physics (solid-state and plasma physics, fluid dynamics and quantum field theory). In this paper, we examine the travelling wave solutions of the Gardner equation and derive the full set of solutions to the corresponding reduced equation in terms of Weierstrass and Jacobi elliptic functions. Then, we use the travelling wave solutions of the focusing mKdV equation and obtain in explicit analytic form exact solutions of a special type of plane curve flow, known as the mKdV flow.

Keywords: Gardner equation, KdV equation, modified KdV equation, travelling wave solutions, Weierstrass and Jacobi elliptic functions, Motion of plane curves, mKdV flow
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INTRODUCTION

The nonlinear evolution partial differential equation

$$u_t + uu_x + \frac{1}{6}\varepsilon^2 u^2 u_x + u_{xxx} = 0, \qquad \varepsilon \in \mathbb{R} \tag{1}$$

usually referred to as the Gardner equation (see [1]), was introduced almost half a century ago in the fist one [2] of a series of works (see also [3]) by Miura, Gardner, Kruskal and coauthors devoted to the study of properties and solutions of the celebrated Korteweg-de Vries (KdV) equation [4]

$$u_t + uu_x + u_{xxx} = 0 \tag{2}$$

and its simplest modification

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{3}$$

currently known as the (focusing) modified Korteweg-de Vries (mKdV) equation. These equations have a great deal in common with the Camassa-Holm equation, but there are significant differences as well [5]. Here and in what follows, the subscripts denote partial differentiations of the dependent variable (unknown function) $u = u(x,t)$ with respect to the indicated independent variables x and t.

In the present paper, by "Gardner equation" we assume a combination of the aforementioned three equations of the form

$$u_t + \alpha_1 u_{xxx} + \alpha_2 u_x + \alpha_3 uu_x + \alpha_4 u^2 u_x = 0, \qquad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}. \tag{4}$$

Thus, by setting $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 0$, $\alpha_4 = (1/6)\varepsilon^2$ in Eq. (4) one obtains the genuine Gardner equation (1), the setting $\alpha_1 = 1$, $\alpha_2 = \alpha_4 = 0$, $\alpha_3 = 1$ yields the KdV equation (2) and, finally, choosing $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$ one gets to the mKdV equation (3). It should be noted that equations of form (4) have attracted a lot of attention in recent years being frequently called "extended KdV (mKdV) equations" (see, *e.g.*, [6, 7, 8]) or "combined KdV–mKdV equations", see, *e.g.*, [9, 10, 11, 12, 13].

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It is noteworthy that, in the most important case $\alpha_1\alpha_4 \neq 0$, the simple invertible transformation of the variables [14]

$$t' = \alpha_1 t, \qquad x' = x - \frac{4\alpha_2\alpha_4 - \alpha_3^2}{4\alpha_4}t, \qquad u = \sqrt{\frac{6\sigma\alpha_1}{\alpha_4}}u' - \frac{\alpha_3}{2\alpha_4}$$

where $\sigma = \mathrm{sign}(\alpha_1/\alpha_4)$, maps Eq. (4), omitting the primes, into the focusing ($\sigma = 1$) or defocussing ($\sigma = -1$) mKdV equation

$$u_t + 6\sigma u^2 u_x + u_{xxx} = 0.$$

Finally, let us remark that the Gardner equation was found to be applicable in various branches of solid-state and plasma physics, fluid dynamics, quantum mechanics and quantum field theory (see, *e.g.*, [6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19] and references therein).

TRAVELLING WAVE SOLUTIONS

Being invariant under any translation of its independent variables x and t, the Gardner equation admits travelling wave solutions of the form

$$u(x,t) = \phi(\xi), \qquad \xi = x - ct, \qquad c \in \mathbb{R} \tag{5}$$

provided that the function $\phi(\xi)$ satisfies the following reduced nonlinear ordinary differential equation

$$\phi'' - \frac{1}{2}c_1 + \frac{\alpha_2 - c}{\alpha_1}\phi + \frac{\alpha_3}{2\alpha_1}\phi^2 + \frac{\alpha_4}{3\alpha_1}\phi^3 = 0, \qquad c_1 \in \mathbb{R}. \tag{6}$$

where the primes denote the derivatives with respect to the variable ξ (here and in what follows we assume $\alpha_1\alpha_4 \neq 0$). Indeed, substituting relations (5) into Eq. (4) and integrating once, we find Eq. (6) in which c_1 is the constant of integration.

Now, multiplying Eq. (6) by ϕ' and integrating again one obtains

$$\left(\phi'\right)^2 - c_2 - c_1\phi + \frac{\alpha_2 - c}{\alpha_1}\phi^2 + \frac{\alpha_3}{3\alpha_1}\phi^3 + \frac{\alpha_4}{6\alpha_1}\phi^4 = 0, \qquad c_2 \in \mathbb{R} \tag{7}$$

where c_2 is the constant of this second integration.

For the purposes of the present study it is convenient to rewrite Eq. (7) in the form

$$\left(\phi'\right)^2 = a_0\phi^4 + 4a_1\phi^3 + 6a_2\phi^2 + 4a_3\phi + a_4 \tag{8}$$

where

$$a_0 = -\frac{\alpha_4}{6\alpha_1}, \qquad a_1 = -\frac{\alpha_3}{12\alpha_1}, \qquad a_2 = \frac{c - \alpha_2}{6\alpha_1}, \qquad a_3 = \frac{1}{4}c_1, \qquad a_4 = c_2.$$

Thus, Eq. (6) is reduced to its first integral (8) whose solutions, therefore, determine entirely all the travelling waves inherent to the Gardner equation. Actually, as noticed in [20], travelling wave solutions of various nonlinear partial differential equations are determined by ordinary differential equations of the form (8) with respective sets of coefficients a_i $(i = 0, \ldots, 4)$. Such an interesting case concerning the so-called Boussinesq paradigm equation [21] will be considered by the present authors in a forthcoming paper.

It should be noted that quite an amount of papers have been published recently (see, *e.g.*, [9, 10, 12, 22, 23, 24, 25]) in which one can find a lot of particular travelling wave solutions to equations of form (4) obtained by applying different, let say, "ansatz methods" (such as the tanh [26] and extended tanh methods [27], Jacobi elliptic function method [28, 29] and many others) to the respective ordinary differential equations, which, as it is shown above, can be written in the form (8). Actually, there is no need to use any "ansatz method" in this case since the general solution of Eq. (8) is readily obtainable in closed form. This will be shown in the next Section where, assuming that the polynomial

$$P(\phi) = a_0\phi^4 + 4a_1\phi^3 + 6a_2\phi^2 + 4a_3\phi + a_4 \tag{9}$$

appearing in the right hand side of Eq. (8) does not have multiple roots, the general solution of Eq. (8) will be given in terms of Weierstrass and Jacobi elliptic functions using the classical result of Weierstrass [30, pp. 4–16] (see also [31, 32]). In fact, it is relatively easy to analyse the cases in which the foregoing polynomial possesses multiple roots and to write down the corresponding general solutions, see [33].

GENERAL EXPRESSIONS FOR THE TRAVELLING WAVE SOLUTIONS

According to Eq. (9), the polynomial $P(\phi)$ appearing in the right-hand side of the first-order nonlinear ordinary differential equation (8) is of fourth degree with respect to the variable ϕ. This allows, following [31, pp. 452–454], to express the general solution of Eq. (8) in the form

$$\phi(\xi) = \rho + \frac{\sqrt{P(\rho)}\wp'(\xi;g_2,g_3) + \frac{1}{2}P_1(\rho)\left(\wp(\xi;g_2,g_3) - \frac{1}{24}P_2(\rho)\right) + \frac{1}{24}P(\rho)P_4(\rho)}{2\left(\wp(\xi;g_2,g_3) - \frac{1}{24}P_2(\rho)\right)^2 - \frac{1}{48}P(\rho)P_4(\rho)} \tag{10}$$

provided that the polynomial $P(\phi)$ does not have multiple roots. Here, ρ is an arbitrary constant, $\wp(\xi;g_2,g_3)$ is the Weierstrass elliptic function, g_2 and g_3 are the invariants of the polynomial $P(\phi)$, which have the form

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4$$

and

$$P_1 = \frac{dP}{d\phi}, \qquad P_2 = \frac{d^2P}{d\phi^2}, \qquad P_4 = \frac{d^4P}{d\phi^4}.$$

If ϕ_0 is a simple root of the polynomial $P(\phi)$, then expression (10) takes the form

$$\phi(\xi) = \phi_0 + \frac{1}{4}\frac{P_1(\phi_0)}{\wp(\xi;g_2,g_3) - \frac{1}{24}P_2(\phi_0)}. \tag{11}$$

The knowledge of the sign of the discriminant

$$\Delta = g_2^3 - 27g_3^2$$

of Weierstrass elliptic function $\wp(\xi;g_2,g_3)$, quartic $P(\phi)$ and cubic

$$R(\xi) = 4\xi^3 - g_2\xi - g_3$$

polynomials allows to express the solution of equation (8) in terms of Jacobi elliptic functions. First, let us remark that the polynomials $P(\phi)$ and $R(\xi)$ do not have multiple roots if and only if $\Delta \neq 0$, see [34, p. 44]. In each such case the Weierstrass elliptic function $\wp(\xi;g_2,g_3)$ appearing in the solution (11) to equation (8) can be expressed, cf. [35, pp. 633, 649–652], as follows:

(i) if $\Delta > 0$, then

$$\wp(\xi;g_2,g_3) = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2\left(\sqrt{e_1 - e_3}\,\xi, m\right)}, \qquad m = \frac{e_2 - e_3}{e_1 - e_3} \tag{12}$$

where $e_1 > e_2 > e_3$ are the roots of the cubic polynomial $R(\xi)$, which, in this case, are real and $\mathrm{sn}(\cdot,\cdot)$ is the Jacobi sine function.

(ii) if $\Delta < 0$, then the polynomial $R(\xi)$ has one real root e_2 as well as a couple of complex conjugated roots e_1, e_3 and

$$\wp(\xi;g_2,g_3) = e_2 + H_2\frac{1 + \mathrm{cn}\left(2\xi\sqrt{H_2}, m\right)}{1 - \mathrm{cn}\left(2\xi\sqrt{H_2}, m\right)}, \qquad m = \frac{1}{2} - \frac{3e_2}{4H_2}, \qquad H_2 = \sqrt{3e_2^2 - \frac{g_2}{4}} \tag{13}$$

where $\mathrm{cn}(\cdot,\cdot)$ is the Jacobi cosine function.

Actually, even if the polynomial $P(\phi)$ has a double root, that is $\Delta = 0$, formula (11) still remains to express the general solution of the corresponding equation of form (8). In each such case, one can use formula (12) or (13) in order to rewrite the solution under question in terms of Jacobi elliptic functions, which now reduce, however, to elementary functions (see [32, 36, 37]).

MOTION OF PLANE CURVES GOVERNED BY THE MKDV CURVE FLOW

Let a regular plane curve parametrized by its arc-length s evolves smoothly and without extension in time t according to the curve flow equation

$$\frac{\partial \mathbf{r}(s,t)}{\partial t} = -\frac{\partial \kappa(s,t)}{\partial s}\mathbf{n}(s,t) - \frac{1}{2}\kappa^2(s,t)\mathbf{t}(s,t) \tag{14}$$

where $\mathbf{r}(s,t)$ is the position vector of the curve, $\kappa(s,t)$ is its curvature, $\mathbf{t}(s,t)$ and $\mathbf{n}(s,t)$ are the unit tangent and inward unit normal vectors to the curve, respectively, defined in the usual way

$$\mathbf{t}(s,t) = \frac{\partial \mathbf{r}(s,t)}{\partial s} = \left(\frac{\partial x(s,t)}{\partial s}, \frac{\partial z(s,t)}{\partial s}\right), \qquad \mathbf{n}(s,t) = \left(-\frac{\partial z(s,t)}{\partial s}, \frac{\partial x(s,t)}{\partial s}\right) \tag{15}$$

where $x(s,t)$ and $z(s,t)$ are the components of the position vector $\mathbf{r}(s,t)$ with respect to a certain rectangular Cartesian coordinate frame XOZ in the Euclidean plane. Note that in terms of the tangent (slope) angle $\varphi(s,t)$ defined as follows

$$\frac{\partial \varphi(s,t)}{\partial s} = \kappa(s,t) \tag{16}$$

one has the expressions

$$\frac{\partial x(s,t)}{\partial s} = \cos\varphi(s,t), \qquad \frac{\partial z(s,t)}{\partial s} = \sin\varphi(s,t). \tag{17}$$

Let us recall also that the tangent $\mathbf{t}(s)$ and normal $\mathbf{n}(s)$ vectors to the curve are related to its curvature $\kappa(s,t)$ through the familiar Frenet-Serret equations

$$\frac{\partial \mathbf{t}(s,t)}{\partial s} = \kappa(s,t)\mathbf{n}(s,t), \qquad \frac{\partial \mathbf{n}(s,t)}{\partial s} = -\kappa(s,t)\mathbf{t}(s,t). \tag{18}$$

Differentiating both sides of Eq. (14) with respect to the variable s and using the Frenet-Serret equations (18) we have

$$\frac{\partial^2 \mathbf{r}(s,t)}{\partial s \partial t} = -\left(\frac{\partial^2 \kappa(s,t)}{\partial s^2} + \frac{1}{2}\kappa^3(s,t)\right)\mathbf{n}(s,t).$$

On account of Eqs. (15) and (17), this relation leads to the following expression for the time derivative of the slope angle

$$\frac{\partial \varphi(s,t)}{\partial t} = -\left(\frac{\partial^2 \kappa(s,t)}{\partial s^2} + \frac{1}{2}\kappa^3(s,t)\right) \tag{19}$$

which, upon a substitution according to Eq. (16) in its right-hand side, can be written in the form

$$\frac{\partial \varphi(s,t)}{\partial t} + \frac{\partial^3 \varphi(s,t)}{\partial s^3} + \frac{1}{2}\left(\frac{\partial \varphi(s,t)}{\partial s}\right)^3 = 0 \tag{20}$$

known as the potential mKdV equation. Differentiating Eq. (20) with respect to the variable s and using again Eq. (16) we arrive at the focusing mKdV equation

$$\frac{\partial \kappa(s,t)}{\partial t} + \frac{\partial^3 \kappa(s,t)}{\partial s^3} + \frac{3}{2}\kappa^2(s,t)\frac{\partial \kappa(s,t)}{\partial s} = 0. \tag{21}$$

This confirms the well-known result of Goldstein and Petrich [38] and Nakayama *et al.* [39] that the plane curve flow of the special form (14) is a mKdV flow, *i.e.*, when a curve evolves according to Eq. (14), the curvature of this curve satisfies the focusing mKdV equation (21).

TRAVELLING WAVE SOLUTIONS OF THE MKDV CURVE FLOW

In what follows, revising the results presented in [40, 41] we will use the travelling wave solutions of the focusing mKdV equation (21) to obtain exact solutions of the curve flow (14) in explicit analytic form. For that purpose, we change to a moving coordinate frame

$$\xi = s - ct, \qquad \tau = t \tag{22}$$

and assume

$$\kappa(\xi,\tau) = \hat{\kappa}(\xi). \tag{23}$$

Then, using the same procedure as in the second Section, from the mKdV equation (21) we find the curvature $\hat{\kappa}(\xi)$ to satisfy the corresponding reduced equation

$$\frac{\mathrm{d}^2\hat{\kappa}}{\mathrm{d}\xi^2} + \frac{1}{2}\hat{\kappa}^3 - c\hat{\kappa} - c_1 = 0 \tag{24}$$

and its first integral

$$\left(\frac{\mathrm{d}\hat{\kappa}}{\mathrm{d}\xi}\right)^2 = \hat{P}(\hat{\kappa}), \qquad \hat{P}(\hat{\kappa}) = -\frac{1}{4}\,\hat{\kappa}^4 + c\hat{\kappa}^2 + 2c_1\hat{\kappa} + c_2 \tag{25}$$

where c_1 and c_2 are the respective constants of integration. Note that Eq. (25) is of the form (8) and hence its general solution is readily expressed in terms of Jacobi elliptic functions through the formulae (11)-(13), see [36, 37].

In terms of the moving coordinate frame (22), the first one of Eqs. (15) and the Frenet-Serret equations (18) read

$$\frac{\partial \mathbf{r}(\xi,\tau)}{\partial \xi} = \mathbf{t}(\xi,\tau), \qquad \frac{\partial \mathbf{t}(\xi,\tau)}{\partial \xi} = \hat{\kappa}(\xi)\mathbf{n}(\xi,\tau), \qquad \frac{\partial \mathbf{n}(\xi,\tau)}{\partial \xi} = -\hat{\kappa}(\xi)\mathbf{t}(\xi,\tau) \tag{26}$$

while the curve flow equation (14) takes the form

$$\frac{\partial \mathbf{r}(\xi,\tau)}{\partial \tau} = -\frac{\mathrm{d}\hat{\kappa}(\xi)}{\mathrm{d}\xi}\mathbf{n}(\xi,\tau) - \frac{1}{2}\left(\hat{\kappa}^2(\xi) - 2c\right)\mathbf{t}(\xi,\tau) \tag{27}$$

the assumption (23) and the first one of Eqs. (26) being taken into account.

In the same frame, expressions (16) and (19) for the derivatives of the tangent angle become

$$\frac{\partial \varphi(\xi,\tau)}{\partial \xi} = \hat{\kappa}(\xi), \qquad \frac{\partial \varphi(\xi,\tau)}{\partial \tau} = -c_1$$

provided that Eqs. (23) and (24) hold. From the above two equations we obtain

$$\varphi(\xi,\tau) = \hat{\varphi}(\xi) - c_1\tau, \qquad \frac{\mathrm{d}\hat{\varphi}(\xi)}{\mathrm{d}\xi} = \hat{\kappa}(\xi). \tag{28}$$

In the case $c_1 \neq 0$, using expressions (28) for the tangent angle one can easily see that

$$\int \cos\left(\hat{\varphi}(\xi) - c_1\tau\right)\mathrm{d}\tau = -\frac{1}{c_1}\sin\left(\hat{\varphi}(\xi) - c_1\tau\right), \qquad \int \sin\left(\hat{\varphi}(\xi) - c_1\tau\right)\mathrm{d}\tau = \frac{1}{c_1}\cos\left(\hat{\varphi}(\xi) - c_1\tau\right)$$

and hence

$$\int \mathbf{n}(\xi,\tau)\mathrm{d}\tau = -\frac{1}{c_1}\mathbf{t}(\xi,\tau), \qquad \int \mathbf{t}(\xi,\tau)\mathrm{d}\tau = \frac{1}{c_1}\mathbf{n}(\xi,\tau)$$

which allows to integrate Eq. (27) with respect to time τ and obtain

$$\mathbf{r}(\xi,\tau) = \frac{1}{c_1}\frac{\mathrm{d}\hat{\kappa}(\xi)}{\mathrm{d}\xi}\mathbf{t}(\xi,\tau) - \frac{1}{2c_1}\left(\hat{\kappa}^2(\xi) - 2c\right)\mathbf{n}(\xi,\tau) + \mathbf{r}_0(\xi) \tag{29}$$

where $\mathbf{r}_0(\xi)$ is an arbitrary vector function. Finally, substituting this in the first one of Eqs. (26) and using the rest of them as well as Eq. (24), we find that the necessary and sufficient condition for a curve of position vector given by Eqs. (29) to evolve according to the curve flow equation (27) is $\mathbf{r}_0$ to be a constant vector.

To summarize, going back to the initial variables s and t, the components of the position vector of a plane curve of curvature $\hat{\kappa}(\xi)$, which evolves according to the curve flow equation (14) can by expressed in the form

$$x(s,t) = \frac{1}{c_1}\frac{\mathrm{d}\hat{\kappa}(\xi)}{\mathrm{d}\xi}\cos\left(\hat{\varphi}(\xi) - c_1 t\right) + \frac{1}{2c_1}\left(\hat{\kappa}^2(\xi) - 2c\right)\sin\left(\hat{\varphi}(\xi) - c_1 t\right) + x_0 \tag{30}$$

$$z(s,t) = \frac{1}{c_1}\frac{\mathrm{d}\hat{\kappa}(\xi)}{\mathrm{d}\xi}\sin\left(\hat{\varphi}(\xi) - c_1 t\right) - \frac{1}{2c_1}\left(\hat{\kappa}^2(\xi) - 2c\right)\cos\left(\hat{\varphi}(\xi) - c_1 t\right) + z_0 \tag{31}$$

where c, c_1, x_0, z_0 are arbitrary constants, $\xi = s - ct$, $\hat{\kappa}(\xi)$ is an arbitrary solution of Eq. (24) and $\mathrm{d}\hat{\varphi}(\xi)/\mathrm{d}\xi = \hat{\kappa}(\xi)$.

The analytic expressions for the aforementioned functions $\hat{\kappa}(\xi)$ and $\hat{\varphi}(\xi)$ are known and can be found, for instance, in the recent papers [33, 36, 37] where the present authors have studied the differential equations (24) and (25) for the curvature $\hat{\kappa}(\xi)$ with the aim to achieve an analytic description of the equilibrium shapes of cylindrical lipid bilayer membranes, elastic rings and tubes under uniform hydrostatic pressure. A summary of these results is presented below.

Depending on the values of the parameters c, c_1 and c_2, there exist two cases in which the polynomial $\hat{P}(\hat{\kappa})$ attains positive values and hence Eq. (25) has real-valued solutions: (I) the polynomial $\hat{P}(\hat{\kappa})$ has two simple real roots $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and a pair of complex conjugate roots $\gamma, \delta \in \mathbb{C}$, $\delta = \bar{\gamma}$; (II) the polynomial $\hat{P}(\hat{\kappa})$ has four simple real roots $\alpha < \beta < \gamma < \delta \in \mathbb{R}$. In the first case, the polynomial $\hat{P}(\hat{\kappa})$ is nonnegative in the interval $\alpha \le \hat{\kappa} \le \beta$, while in the second one, it is nonnegative in the intervals $\alpha \le \hat{\kappa} \le \beta$ and $\gamma \le \hat{\kappa} \le \delta$. It should be noted that the roots of the polynomial $\hat{P}(\hat{\kappa})$ can be expressed explicitly through its coefficients and vice versa, see the Appendix.

Let the parameters c, c_1 and c_2 be such that the polynomial $\hat{P}(\hat{\kappa})$ has roots as in case (I), namely, two of them are real ($\alpha < \beta$) and the other two constitute a complex conjugate pair which, in view of relation (32), can be written in the form

$$\gamma = -\frac{\alpha+\beta}{2} + \mathrm{i}\eta, \qquad \delta = -\frac{\alpha+\beta}{2} - \mathrm{i}\eta$$

where η is a nonnegative real number. In this case, equation (25) has periodic solutions if $\eta \neq 0$ or $\eta = 0$ and $(3\alpha+\beta)(\alpha+3\beta) > 0$, see [36, Theorem 1].

Let $\eta \neq 0$ and hence the roots of the polynomial $\hat{P}(\hat{\kappa})$ are simple. Denote

$$\lambda_1 = \frac{1}{4}\sqrt{AB}, \qquad k_1 = \sqrt{\frac{1}{2} - \frac{4\eta^2 + (3\alpha+\beta)(\alpha+3\beta)}{2AB}}$$

where

$$A = \sqrt{4\eta^2 + (3\alpha+\beta)^2}, \qquad B = \sqrt{4\eta^2 + (\alpha+3\beta)^2}.$$

Evidently, $A > 0$, $B > 0$, $\lambda_1 > 0$ and $0 < k_1 < 1$. In this case, each solution of Eq. (25) can be expressed by the function

$$\hat{\kappa}_1(\xi) = \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\,\mathrm{cn}(\lambda_1\xi, k_1)}{(A+B) - (A-B)\,\mathrm{cn}(\lambda_1\xi, k_1)}$$

which takes real values for each $\xi \in \mathbb{R}$ and is periodic with least period $T_1 = (4/\lambda_1)\mathrm{K}(k_1)$ due to the periodicity of the Jacobi function $\mathrm{cn}(\lambda_1\xi, k_1)$. Here, $\mathrm{K}(\cdot)$ denotes the complete elliptic integral of the first kind. The corresponding slope angle (28) can be written in the form

$$\begin{aligned}\hat{\varphi}_1(\xi) \;=\; & \frac{A\beta - B\alpha}{A-B}\xi + \frac{\alpha-\beta}{2\lambda_1\sqrt{k_1^2 + \frac{(A-B)^2}{4AB}}}\arctan\left(\sqrt{k_1^2 + \frac{(A-B)^2}{4AB}}\,\frac{\mathrm{sn}(\lambda_1\xi, k_1)}{\mathrm{dn}(\lambda_1\xi, k_1)}\right) \\ & + \frac{(A+B)(\alpha-\beta)}{2\lambda_1(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, \mathrm{am}(\lambda_1\xi, k_1), k_1\right)\end{aligned}$$

where $\Pi(\cdot,\cdot,\cdot)$ denotes the incomplete elliptic integral of the third kind.

Now, let $\eta = 0$ and $(3\alpha+\beta)(\alpha+3\beta) > 0$. Then, the polynomial $\hat{P}(\hat{\kappa})$ has one double and two simple real roots. The curvature and the slope angle (28) are expressed in terms of elementary functions as follows

$$\begin{aligned}\hat{\kappa}_2(\xi) \;&=\; \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\cos(\lambda_1\xi)}{(A+B) - (A-B)\cos(\lambda_1\xi)} \\ \hat{\varphi}_2(s) \;&=\; \frac{A\beta - B\alpha}{A-B}\xi + \frac{8(\alpha-\beta)}{A-B}\arctan\left(\sqrt{\frac{A}{B}}\tan\left(\frac{1}{2}\lambda_1\xi\right)\right).\end{aligned}$$

Let the parameters c, c_1 and c_2 be such that the polynomial $\hat{P}(\hat{\kappa})$ has roots as in case (II), that is $\alpha < \beta < \gamma < \delta \in \mathbb{R}$. Denote

$$\lambda_2 = \frac{1}{4}\sqrt{(\gamma-\alpha)(\delta-\beta)}, \qquad k_2 = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)}}.$$

Since $\gamma-\alpha > \beta-\alpha > 0$ and $\delta-\beta > \delta-\gamma > 0$, it is seen that $\lambda_2 > 0$ and $0 < k_2 < 1$. In this case, each solution of Eq. (25) can be expressed by one of the following functions

$$\hat{\kappa}_3(\xi) = \delta - \frac{(\delta-\alpha)(\delta-\beta)}{(\delta-\beta)+(\beta-\alpha)\mathrm{sn}^2(\lambda_2\xi,k_2)}, \qquad \hat{\kappa}_4(\xi) = \beta + \frac{(\gamma-\beta)(\delta-\beta)}{(\delta-\beta)-(\delta-\gamma)\mathrm{sn}^2(\lambda_2\xi,k_2)}$$

see [36, Theorem 2]. The functions $\hat{\kappa}_3(\xi)$ and $\hat{\kappa}_4(\xi)$ take real values for each $\xi \in \mathbb{R}$ and are periodic with least period $T_2 = (2/\lambda_2)\mathrm{K}(k_2)$ because of the periodicity of the function $\mathrm{sn}^2(\lambda_2\xi,k_2)$. Their indefinite integrals (28) can be written respectively as

$$\hat{\varphi}_3(\xi) = \delta\xi - \frac{\delta-\alpha}{\lambda_2}\Pi\left(\frac{\beta-\alpha}{\beta-\delta},\mathrm{am}(\lambda_2\xi,k_2),k_2\right), \qquad \hat{\varphi}_4(\xi) = \beta\xi - \frac{\beta-\gamma}{\lambda_2}\Pi\left(\frac{\delta-\gamma}{\delta-\beta},\mathrm{am}(\lambda_2\xi,k_2),k_2\right).$$

It should be remarked, that the indefinite integrals $\hat{\varphi}_j(\xi)$, $j = 1,\ldots,4$, of the foregoing solutions $\hat{\kappa}_j(\xi)$ of Eq. (25) are chosen so that $\hat{\varphi}_j(0) = 0$. Moreover, $\hat{\kappa}_j(0)$ always coincides with a certain root of the polynomial $\hat{P}(\hat{\kappa})$ (actually, $\hat{\kappa}_j(0) = \alpha$ for $j = 1,2,3$ and $\hat{\kappa}_4(0) = \gamma$) and hence $\mathrm{d}\hat{\kappa}/\mathrm{d}\xi = 0$ at $\xi = 0$, according to Eq. (25).

The evolution of a closed plane curve obtained through the expressions (30) and (31) for the coordinates of its position vectors is depicted in Figure 1.

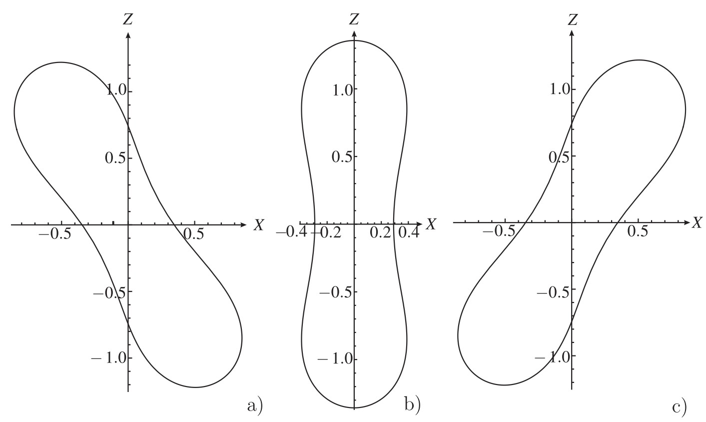

FIGURE 1. Evolution of a closed plane curves of curvature $\hat{\kappa}(\xi) = \hat{\kappa}_1(\xi)$ and tangent angle $\hat{\varphi}(\xi) = \hat{\varphi}_1(\xi)$ drawn via the expressions (30) and (31) with $x_0 = z_0 = 0$, $c = -1.018$, $c_1 = 4$ and $c_2 = 4.66383$ at time (a) $t = \pi/12$, (b) $t = \pi/8$, (c) $t = \pi/6$

APPENDIX

After some standard algebraic manipulations (see, *e.g.*, [42]), one can find the following expressions for the roots of the polynomial $\hat{P}(\hat{\kappa})$

$$-\sqrt{\omega} \pm \sqrt{2c - \frac{2c_1}{\sqrt{\omega}} - \omega}, \qquad \sqrt{\omega} \pm \sqrt{2c + \frac{2c_1}{\sqrt{\omega}} - \omega}$$

where

$$\omega = \frac{(2c + \sqrt[3]{2^2 3\left(3^2c_1^2 + \sqrt{\chi}\right) - 2^3c\left(c^2+3^2c_2\right)})^2 - 2^2 3c_2}{6\sqrt[3]{2^2 3\left(3^2c_1^2 + \sqrt{\chi}\right) - 2^3c\left(c^2+3^2c_2\right)}}, \qquad \chi = 2^2 3c_2\left[\left(c^2+c_2\right)^2 - 3^2c_1^2c\right] - 3c_1^2\left(2^2c^3 - 3^3c_1^2\right).$$

Then, one can denote properly each of the above expressions for the roots in accordance with the notation introduced in the cases (I) and (II), respectively. Simultaneously, by Vieta's formulas one obtains

$$\alpha+\beta+\gamma+\delta=0 \tag{32}$$

due to the absence of a term with $\hat{\kappa}^3$ in the polynomial $\hat{P}(\hat{\kappa})$, see Eq. (25), and

$$c=\frac{1}{4}\left(\alpha^2+\beta^2+\gamma^2+\alpha\beta+\alpha\gamma+\beta\gamma\right), \qquad c_1=-\frac{1}{8}(\alpha+\beta)(\alpha+\gamma)(\beta+\gamma), \qquad c_2=\frac{1}{4}\alpha\beta\gamma(\alpha+\beta+\gamma).$$

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