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On the Exponents of Some 4×4 Matrices

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Abstract

Here we derive a formula for the exponential of any 4×4 matrix which belongs to one of the Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(3,1)$ and $\mathfrak{sp}(4,\mathbb{R})$.
The approach which we follow is based on the Hamilton-Cayley theorem, namely, the important moment in all our considerations are the respective characteristic polynomials of the above matrices.

Introduction

Many mathematical models of processes in Physics, Biology and Chemistry are based on systems of linear, ordinary differential equations with constant coefficients. For example, by rewriting the classical Lorentz force law equation in the relativistic form, the motion of a charged particle in a constant electromagnetic field can be described by a system of four linear differential equations - the so called Lorentz equations

$$\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = a\mathcal{F}^\alpha_\beta U^\beta, \qquad \alpha, \beta = x, y, z, t. \tag{1}$$

where a is a real constant, U denotes the particle's four-velocity (column) vector with respect to the fixed inertial system

$$U = {}^t(U^x, U^y, U^z, U^t), \qquad {}^t = \text{transpose} \tag{2}$$

and

$$\mathcal{F} = \begin{bmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ E_1 & E_2 & E_3 & 0 \end{bmatrix}. \tag{3}$$

As one can see $\mathcal{F}$ is an element in the Lie algebra $\mathfrak{so}(3,1)$, which is the Lie algebra of the Lorentz group SO$(3,1)$. The general solution of Lorentz equation is

$$U(\tau) = \text{Exp}(a\mathcal{F}\tau)U(0) \tag{4}$$

where Exp is the exponential map from the Lie algebra $\mathfrak{so}(3,1)$ to Lorentz group SO$(3,1)$ and $U(0)$ is the initial value of 4-velocity vector (at the proper time $\tau = 0$).

So finding of the particle's trajectory is equivalent to finding of formula for the exponential map from $\mathfrak{so}(3,1)$ to $SO(3,1)$.

Another physical models lead to the necessity of finding formulas for the exponential maps from various Lie algebras to the corresponding Lie groups.

A formula for the exponential map for Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$ and $\mathfrak{sp}(4)$ is derived.

The Exponential Map

Let G be a finite dimensional Lie Group with unit e and $\mathfrak{g}$ - its Lie algebra. The exponential map from $\mathfrak{g}$ to G is defined as follows

$$\begin{aligned} \mathrm{Exp} &: \mathfrak{g} \to G \\ \mathrm{Exp}(X) &= \gamma_X(1) \end{aligned} \tag{5}$$

where γ_X is the integral curve of the left-invariant vector field $X \in \mathfrak{g}$ with initial condition $\gamma_X(0) = e$.

It is well known that if G is a group of $n \times n$ real or complex matrices then

$$\mathrm{Exp}(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \tag{6}$$

where X is $n \times n$ matrix from the Lie algebra $\mathfrak{g}$.

Consequently, since the groups which we will consider: SO(4), SO(3, 1), SO(2, 2), Sp(4) are matrix Lie groups, we have to calculate the above power series for any matrix X belonging to one of Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$ and $\mathfrak{sp}(4)$.

Actually, these calculations can be replaced by a single one by noticing that all 4×4 matrices in the above listed algebras share the same characteristic polynomial of the following special type

$$f(z) = z^4 - bz^2 - a. \tag{7}$$

Calculation of Power Series

Let A be some 4×4 matrix with characteristic polynomial

$$f(z) = z^4 - bz^2 - a. \tag{8}$$

Cayley-Hamilton's theorem implies

$$A^4 = a\mathrm{I}_4 + bA^2. \tag{9}$$

Direct consequence of this equality is

$$\mathrm{Exp}(A) = \mathrm{I}_4 + A + \frac{A^2}{2} + \frac{A^3}{6} \qquad a = b = 0. \tag{10}$$

So we may consider

$$a \neq 0 \qquad \text{or} \qquad b \neq 0.$$

First we write the equality (9) in the following manner

$$A^4 = uv\mathrm{I}_4 + (u - v)A^2 \tag{11}$$

where u and v are new parameters, determined by the system

$$u - v = b \qquad uv = a. \tag{12}$$

Straightforward consequence is

$$(u + v)^2 = b^2 + 4a. \tag{13}$$

We derive two formulas: one for the case $b^2 + 4a \neq 0$, and the other for $b^2 + 4a = 0$.

The case $b^2 + 4a \neq 0$.

Using

$$A^4 = uv\mathrm{I}_4 + (u - v)A^2 \tag{14}$$

by finite induction one can prove, that for each $n \geq 0$

$$\begin{aligned} (u+v)A^{2n} = \ & (v\ u^n + (-1)^n u\ v^n)\mathrm{I}_4 \\ & +(u^n + (-1)^{n+1} v^n)A^2 \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} (u+v)\sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!} = \ & \left(v\sum_{n=0}^{\infty}\frac{u^n}{(2n)!} + u\sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n)!}\right)\mathrm{I}_4 \\ & + \left(\sum_{n=0}^{\infty}\frac{u^n}{(2n)!} - \sum_{n=0}^{\infty}\frac{(-1)^n v^n}{(2n)!}\right)A^2 \end{aligned}$$

$$= \left(v\cosh\sqrt{u} + u\cos\sqrt{v}\right)\mathrm{I}_4 + \left(\cosh\sqrt{u} - \cos\sqrt{v}\right)A^2. \tag{16}$$

Using (15) again, by similar considerations is obtained

$$(u+v)\sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!} = \left(v\frac{\sinh\sqrt{u}}{\sqrt{u}}+u\frac{\sin\sqrt{v}}{\sqrt{v}}\right)A+\left(\frac{\sinh\sqrt{u}}{\sqrt{u}}-\frac{\sin\sqrt{v}}{\sqrt{v}}\right)A^{3}. \tag{17}$$

Now introducing functions

$$f_0(u,v) = \frac{v \cosh \sqrt{u} + u \cos \sqrt{v}}{u+v}$$

$$f_1(u,v) = \frac{v\frac{\sinh \sqrt{u}}{\sqrt{u}} + u\frac{\sin \sqrt{v}}{\sqrt{v}}}{u+v}$$

$$f_2(u,v) = \frac{\cosh \sqrt{u} - \cos \sqrt{v}}{u+v}$$

$$f_3(u,v) = \frac{\frac{\sinh \sqrt{u}}{\sqrt{u}} - \frac{\sin \sqrt{v}}{\sqrt{v}}}{u+v}$$

allows to infer

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{A^{2n}}{(2n)!} &= f_0(u,v)\mathrm{I}_4 + f_2(u,v)A^2 \\
\sum_{n=0}^{\infty} \frac{A^{2n+1}}{(2n+1)!} &= f_1(u,v)A + f_3(u,v)A^3
\end{aligned}
\tag{18}
$$

which implies

$$
\begin{aligned}
\mathrm{Exp}(A) = \; & f_0(u,v)\mathrm{I}_4 + f_1(u,v)A \\
& + f_2(u,v)A^2 + f_3(u,v)A^3.
\end{aligned}
\tag{19}
$$

The case $b^2 + 4a = 0$.

If this equality is satisfied then

$$A^4 - bA^2 - a\mathrm{I}_4 = \left(A^2 - \frac{b}{2}\mathrm{I}_4\right)^2$$

$$= \left(A - \sqrt{\frac{b}{2}}\mathrm{I}_4\right)^2 \left(A + \sqrt{\frac{b}{2}}\mathrm{I}_4\right)^2.$$

Now the Cayley-Hamilton's theorem implies

$$(A - \rho\mathrm{I}_4)^2 (A + \rho\mathrm{I}_4)^2 = 0. \tag{20}$$

where

$$\sqrt{\frac{b}{2}} = \rho. \tag{21}$$

Equality (18) brings to

$$\begin{aligned}
&\mathrm{Exp}\,(A-\rho\mathrm{I}_4)\,(A+\rho\mathrm{I}_4)^2 \\
&= [\mathrm{I}_4+(A-\rho\mathrm{I}_4)]\,(A+\rho\mathrm{I}_4)^2 \\
&= [A+(1-\rho)\,\mathrm{I}_4]\,(A+\rho\mathrm{I}_4)^2 \,.
\end{aligned} \tag{22}$$

Now since matrices A and $\rho\mathrm{I}_4$ commute

$$\begin{aligned}
&\mathrm{Exp}(A)\,(A+\rho\mathrm{I}_4)^2 \\
&= \exp\,(\rho)\,[A+(1-\rho)\,\mathrm{I}_4]\,(A+\rho\mathrm{I}_4)^2 \,.
\end{aligned} \tag{23}$$

Through similar considerations one can prove that

$$\begin{aligned}
&\mathrm{Exp}(A)\,(A-\rho\mathrm{I}_4)^2 \\
&= \exp\,(-\rho)\,[A+(1+\rho)\,\mathrm{I}_4]\,(A-\rho\mathrm{I}_4)^2 \,.
\end{aligned} \tag{24}$$

Subtraction of last two equations gives

$$\begin{aligned} 4\rho \mathrm{Exp}(A)A = \; & \exp(\rho)\left[A + (1-\rho)\,\mathrm{I}_4\right](A+\rho \mathrm{I}_4)^2 \\ & - \exp(-\rho)\left[A + (1+\rho)\,\mathrm{I}_4\right](A-\rho \mathrm{I}_4)^2 \end{aligned} \tag{25}$$

The determinant of the matrix A is $-a$, hence A is invertible (because $a \neq 0$). Now after some calculations one may easily obtain the following final formula

$$\mathrm{Exp}(A) = g_0(\rho)A^{-1} + g_1(\rho)\mathrm{I}_4 + g_2(\rho)A + g_3(\rho)A^2 \tag{26}$$

where

$$g_0(\rho) = \frac{\rho \sinh \rho - \rho^2 \cosh \rho}{2}$$

$$g_1(\rho) = \frac{2 \cosh \rho - \rho \sinh \rho}{2}$$

$$g_2(\rho) = \frac{\sinh \rho + \rho \cosh \rho}{2\rho}$$

$$g_3(\rho) = \frac{\sinh \rho}{2\rho}.$$

The parameters a and b for the Lie algebra $\mathfrak{so}(4)$

The Lie algebra $\mathfrak{so}(4)$ is generated by the skew-symmetric real matrices

$$A = \left\{ \begin{bmatrix} 0 & -x_1 & x_2 & -x_4 \\ x_1 & 0 & -x_3 & -x_5 \\ -x_2 & x_3 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{bmatrix}; \; x_i \in \mathbb{R} \right\}$$

and the parameters a and b are determined by the coordinates x_i $i = 1, \ldots, 6$ through the following formulas

$$\begin{aligned} a &= -(x_3x_4 + x_2x_5 + x_1x_6)^2 \\ b &= -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2. \end{aligned} \tag{27}$$

The parameters a and b for the Lie algebra $\mathfrak{so}(3,1)$

The Lie algebra of the Lorentz group $\mathfrak{so}(3,1)$ is represented by the the real 4×4 matrices of the form

$$\mathfrak{so}(3,1) = \left\{ \begin{bmatrix} 0 & -x_1 & x_2 & x_4 \\ x_1 & 0 & -x_3 & x_5 \\ -x_2 & x_3 & 0 & x_6 \\ x_4 & x_5 & x_6 & 0 \end{bmatrix} ; \; x_i \in \mathbb{R} \right\}.$$

Here the parameters a and b are

$$\begin{aligned} a &= (x_3 x_4 + x_2 x_5 + x_1 x_6)^2 \\ b &= -x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 + x_6^2. \end{aligned} \tag{28}$$

The parameters a and b for the Lie algebra $\mathfrak{so}(2,2)$

The general elements of the Lie algebra $\mathfrak{so}(2,2)$ are represented by the matrices of the form

$$\mathfrak{so}(2,2) = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_4 \\ -x_1 & 0 & x_3 & x_5 \\ x_2 & x_3 & 0 & x_6 \\ x_4 & x_5 & -x_6 & 0 \end{bmatrix}; \; x_i \in \mathbb{R} \right\}$$

and the corresponding parameters a and b are given by the expressions

$$\begin{aligned} a &= -(-x_3x_4 + x_2x_5 + x_1x_6)^2 \\ b &= -x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2. \end{aligned} \tag{29}$$

The parameters a and b for the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$

Finally, the Lie algebra elements of $\mathfrak{sp}(4,\mathbb{R})$ are of the form

$$\mathfrak{sp}(4,\mathbb{R}) = \left\{ \begin{bmatrix} x_1 & x_2 & x_5 & x_6 \\ x_3 & x_4 & x_6 & x_7 \\ x_8 & x_9 & -x_1 & -x_3 \\ x_9 & x_{10} & -x_2 & -x_4 \end{bmatrix} ;\; x_i \in \mathbb{R} \right\}$$

and their parameters a and b are respectively

$$\begin{aligned} a \;=\; & -x_1^2x_4^2 - x_2^2x_3^2 - x_6^2x_9^2 - x_1^2x_7x_{10} - x_3^2x_5x_{10} \\ & -x_2^2x_7x_8 - x_4^2x_5 + x_6^2x_8x_{10} + x_9^2x_5x_7x_8 \\ & +2x_1x_2x_3x_4 - 2x_1x_4x_6x_9 + 2x_1x_2x_7x_9 \\ & +2x_1x_3x_6x_{10} + 2x_2x_4x_6x_8 - 2x_2x_3x_6x_9 \\ & +2x_3x_4x_5x_9 - x_5x_7x_8x_{10} \end{aligned} \tag{30}$$

$$b \;=\; x_1^2 + x_4^2 + 2x_2x_3 + 2x_6x_9 + x_5x_8 + x_7x_{10}.$$

The coefficients a and b for all Lie algebras listed above are obtained by means of the Leverrier-Faddeev Characteristic Polynomial Algorithm, which is realized as a separate computer program in "Mathematica" language.