



## GENERALIZATION OF HADAMARD'S LAPLACE EIGENVALUE FORMULA TO DEFORMING MANIFOLDS

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**Abstract.** For a Euclidean domain with a moving boundary, Hadamard's formula relates the rate of change of the Laplace eigenvalues to the normal velocity of the boundary. We generalize Hadamard's formula to deforming Riemannian manifolds with contour boundary moving in a compatible manner. Our analysis finds direct applications in the dynamics of fluid films. The spectrum of the surface Laplacian describes the frequencies of normal oscillations of the film's surface as well as tangential oscillations in thickness.

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### 1. Introduction

How do the eigenvalues of the Laplace operator depend on the shape of the domain? This question was originally posed by Hadamard.

In 1908, having planted the seeds of what we now call the calculus of moving surfaces, Hadamard established an expression for the rate of change in Laplace eigenvalues for a deforming domain  $\Omega$  – thereby collecting the first fruits of the new calculus [3]. This subject has been an area of active research. The reader will find excellent reviews in [4] and [12] along with many useful references therein.

Hadamard's formula applies to the Laplace operator on Euclidean domains with deforming boundaries. This goal of this paper is to establish analogous results for the surface Laplacian on deforming Riemannian manifolds with moving contour boundaries.

The surface Laplacian  $\nabla_\alpha \nabla^\alpha$  finds a great number of applications. In [1], we present exact Hamiltonian equation for the dynamics of fluid films. We show that the equation that governs small normal oscillations about an equilibrium configuration is  $u_{tt} = c^2 (\nabla_\alpha \nabla^\alpha - 2K) u$ , where  $K$  is Gaussian curvature. Furthermore, small oscillations in thickness  $\rho$  are governed by the surface wave equation  $\rho_{tt} = c^2 \nabla_\alpha \nabla^\alpha \rho$ . These examples demonstrate applications of the surface Laplacian and illustrate the immediate relevance of its spectrum.