JGSP **2021** (2011) 95–98

JOURNAL OF
Geometry and Symmetry
in Physics

BOOK REVIEW

Elementary Differential Geometry, by Andrew Pressley, Springer, London 2010, xi+473 pp., ISBN 978-1-84882-890-2.

The book is devoted to the differential geometry of curves and surfaces in the three-dimensional Euclidean space and consists of thirteen chapters.

First Chapter - *Curves in the Plane and in Space*, is split into five sections. At first two types of curves are discussed - level curves and parameterized curves, as well the relation between them. For a smooth parameterized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n, n \geqq 2, -\infty \leqq \alpha < \beta \leqq \infty$, the author introduces the notions: the tangent vector $\dot{\gamma}(t)$ at the point $\gamma(t)$, an arc length $s(t) = \int_{t_0}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$ starting at the point $\gamma(t_0)$, the speed $||\dot{\gamma}(t)||$ of the point $\gamma(t)$ and reparametrization. The following objects are also defined: a unit-speed, regular, periodic and closed curves. It is proved that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

Chapter two - *How Much Does a Curve Curve*, is structured into three sections. This chapter contains the following themes of a regular curve: curvature κ (measuring the extent to which a curve which is not contained in a straight line curve), torsion τ (measuring the extent to which a curve in $\mathbb{R}^3$ is not contained in a plane curve in $\mathbb{R}^n, n \geqq 2$), signed unit, normal n_s and signed curvature κ_s of a plane curve, total signed curvature $\int_0^l \kappa_s \mathrm{d}s$ of a closed curve of length l, a turning angle $\varphi(s)$ of a plane curve, Frenet-Serret equations of a curve in $\mathbb{R}^3$. It is proved that: $\kappa_s = \mathrm{d}\varphi/\mathrm{d}s$ (i.e., the signed curvature is the rate of which the tangent vector of the curve rotates) and the theorem that the curvature κ and the torsion τ (for a curve in $\mathbb{R}^3$) determine the curve up to a motion in the space.

Chapter three - *Global Properties of Curves*, is split into three sections. Here are presented (without complete proofs) the following global results for a simple closed curve γ - the Hopf's Umlaufsatz that the total signed curvature of γ is $\pm 2\pi$, the isoperimetric inequali $A(\gamma) \leqq 1/4\pi l(\gamma)^2$, where $A(\gamma)$ is the area contained by γ and $l(\gamma)$ is the length of γ and the Four Vertex Theorem.

Chapter four - *Surfaces in Three Dimensions*, is structured into five sections. A subset S of $\mathbb{R}^3$ is called a surface if for every point $p \in S$ there is an open set U

in $\mathbb{R}^2$, an open set W in $\mathbb{R}^3, p \in W$ and a homeomorphism: $\sigma : U \to S \cap W$. The homeomorphism σ is called a surface patch or a parametrization of the open subset $S \cap W$ of S. A collection of such surface patches whose images cover the whole of S is called an atlas of S. If σ is smooth and the vectors σ_u and σ_v are linearly independent the surface patch $\sigma : U \to S \cap W$ is called regular. The surface S is called smooth if for any point $p \in S$ there is a regular patch σ such that $p \in \sigma(U)$. The following themes are discussed: a reparametrization of a surface patch given by a smooth map $f : S_1 \to S_2$, where S_1 and S_2 are smooth surfaces, the tangent space T_pS of S at the point p, the derivative D_pf of f at p, the normals and orientability.

Chapter five - *Examples of Surfaces*, is split into six sections. Here are given general conditions under which a level surface is a smooth surface, and there are considered the quadric surfaces, the ruled surfaces, the surfaces of revolution, the compact surfaces and the triply orthogonal systems of surfaces.

Chapter six - *The First Fundamental Form*, is split into five sections. The following themes are considered: length of curves on surfaces, isometries of surfaces, conformal mappings of surfaces, equiareal map and the theorem of Archimedes, spherical geometry. Besides, here are given: necessary and sufficient conditions for a smooth map $f : S_1 \to S_2$ to be a local isometry (correspondingly conformal equiareal map), the spherical distance between two points of the sphere and some theorems for spherical triangles.

Chapter seven - *Curvature of Surfaces*, consist of four sections. In this chapter several approaches to the problem of measuring how "curved" a surface is are discussed. The chapter contains the following themes: the second fundamental form, Gauss and Weingarten maps, normal and geodesic curvatures, parallel transport and covariant derivative. Here are proved necessary and sufficient conditions for a tangent vector field to be parallel along a curve on a surface.

Chapter eight - *Gaussian, Mean and Principal Curvatures*, is split into following six sections: Gaussian and mean curvatures, Principal curvatures of a surface, Surfaces of constant Gaussian curvature, Flat surfaces, Surfaces of constant mean curvature, Gaussian curvature of compact surfaces.

Chapter nine - *Geodesics*, Consists of Five Sections. Here are given the geodesic equations and are studied some basic properties of a geodesic and particulary geodesics on surfaces of revolution, geodesics as shortest paths and geodesic coordinates.

Chapter ten - *Gauss' Theorem Egregium*, is structured into four sections and contains: the Gauss and Codazzi-Mainardi equations, the formulation of the theorem that the first and second fundamental forms determine a surface patch up to a direct isometry of $\mathbb{R}^3$, the Gauss' remarkable theorem, that the Gaussian curvature of a surface is - preserved by local isometries, some results of the surfaces of constant Gaussian curvature, geodesic mappings.

Chapter eleven - *Hyperbolic Geometry*, consists of five sections. Here the author had considered the Upper half-plane model, Poincare disc model and Beltrami-Klein model of the hyperbolic geometry and present various results about hyperbolic lines, hyperbolic angles, hyperbolic distance, hyperbolic area, hyperbolic triangles and hyperbolic parallels.

Chapter twelve - *Minimal Surfaces*, is split into five sections. Note that a surface is called minimal if its mean curvature is zero everywhere. The chapter begins with Plateau's Problem - the problem of finding a surface of minimal area with a fixed curve as its boundary. It is proved that if a surface S is a solution of the Plateau's Problem, then S is a minimal surface. The illustrations of minimal surfaces include a catenoid, a helicoid, Enneper's surface and Scherk's surface. Gauss map of a minimal surface, conformal parametrization of minimal surfaces, connection between minimal surfaces and holomorphic functions, the Weierstrass representation are also discussed in some depth.

Chapter thirteen is devoted to the most beautiful and profound result in the theory of surfaces - *The Gauss-Bonnet Theorem.* This chapter is split into eight sections. The theorem is proved initially for a simple closed curve, then for a curvilinear polygon, and finally for a compact surface. The last case is the most important version of the Gauss-Bonnet theorem, which establishes the relation between the average of the Gaussian curvature K over the whole surface S and the Euler number χ of S. The chapter ends with applications of the Gauss-Bonnet theorem to the problems of the map coloring, the holonomy around a unit-speed closed curve on a surface, the number of the stationary points of a smooth, vector field on a compact surface, the non-degenerate critical points of a smooth function on a compact surface.

There are three Appendixes to the book where are collected some basic results of the linear algebra, about isometries of $\mathbb{R}^3$ and the Möbius transformations that are used in the book.

Hints to selected exercises and solutions to all exercises included in the book are given at the end as well. There are also index pages.

A distinctive feature of the book is the large collection of themes and exercises (over 250), which make it interesting and useful. As a whole the hook is well written and very well ordered. There are many examples and illustrations, which make it pleasant for reading.

This book is recommended to undergraduate and to graduate students, specializing in geometry, and to all interested scientists. For further reading we suggest [1–4].

References

[1] Doss-Bachelet C., Françoise J.-P. and Piquet C., *Géométrie Differentielle*, Ellipses, Paris, 2000.

[2] Gray A., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd Ed., CRC Press, Boca Raton, FL, 1998.

[3] Oprea J., *Differential Geometry and Its Applications*, Mathematical Association of America, Washington D. C., 2007.

[4] Rovenski V., *Geometry of Curves and Surfaces with Maple*, Birkhäuser, Boston, 2000.

Ivanka Ivanova-Karatopraklieva
Faculty of mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
1129 Sofia, BULGARIA
E-mail address: karatopraklieva@fmi.uni-sofia.bg