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FORCE FREE MÖBIUS MOTIONS OF THE CIRCLE

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Abstract. Let $\mathcal{M}$ be the Lie group of Möbius transformations of the circle. Suppose that the circle has initially a homogeneous distribution of mass and that the particles are allowed to move only in such a way that two configurations differ in an element of $\mathcal{M}$. We describe all force free Möbius motions, that is, those curves in $\mathcal{M}$ which are critical points of the kinetic energy. The main tool is a Riemannian metric on $\mathcal{M}$ which turns out to be not complete (in particular not invariant, as happens with non-rigid motions) given by the kinetic energy.

1. Introduction

In the spirit of the classical description of the force free motions of a rigid body in Euclidean space using an invariant metric on $\mathrm{SO}(3)$ [1, Appendix 2], the second author defined in [4] an appropriate metric on the Lorenz group $\mathrm{SO}_o(n+1,1)$ to study force free conformal motions of the sphere $\mathbb{S}^n$, obtaining a few explicit ones (only through the identity and those which can be described using the Lie structure of the configuration space). In this note, in the particular case $n = 1$, that is, Möbius motions of the circle, we obtain all force free motions.

This is an example of a situation in which using concepts of Physics one can state and solve a problem in Differential Geometry (see for instance [2, 3, 6]).

Notice that the canonical action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \cong \mathbb{S}^1$ is equivalent to the action of the group of Möbius transformations on the circle. Then, the results presented here, up to a double covering, also extend the case $n = 1$ of [5], where force free projective motions of the sphere $\mathbb{S}^n$ were studied.

This note, as well as [4, 5], is weakly related with mass transportation [7]. In our situation, the set of admitted mass distributions is finite dimensional, and also the allowed transport maps are very particular.