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A RELATION BETWEEN THE CYLINDRIC FLUID MEMBRANES AND THE MOTIONS OF PLANAR CURVES

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Abstract. We observe a relation between the mKdV equation and the cylindrical equilibrium shapes of fluid membranes. In our setup mKdV arises from the study of the evolution of planar curves.

1. Introduction

The goal of this paper is to unify and extend the results presented in [5] and [8]. It also shows a connection between two problems that appear unrelated.

The first problem comes from the study of equilibrium shapes of fluid membranes. One starts with a functional proposed by Helfrich (see [2], [8]) and studies the corresponding Euler-Lagrange equation. The equilibrium shapes are given as the extremals of the functional

$$\mathcal{F} = \frac{k_c}{2} \int_S (2H + \mathbb{h})^2 \mathrm{d}A + k_G \int_S K \mathrm{d}A + \lambda \int_S \mathrm{d}A + p \int \mathrm{d}V. \tag{1}$$

Notice that $\mathcal{F}$ is closely related to the Willmore energy functional. The Euler-Lagrange equation associated with $\mathcal{F}$ is as follows

$$2k_c \Delta_S H + k_c(2H + \mathbb{h})(2H^2 - \mathbb{h}H - 2K) - 2\lambda H + p = 0. \tag{2}$$

Here H and K are the mean and Gauss curvatures respectively, k_c and k_G - bending and Gaussian constant rigidity of the membrane, $\mathbb{h}$ is spontaneous curvature constant, p and λ - Lagrange multipliers corresponding to fixed volume and total membrane area and Δ_S is the surface Laplacian on the interface of the membrane. The nature of this equation is complex as it involves the surface Laplacian of the mean curvature which makes it a fourth-order non-linear PDE. However, as always, the symmetry of the problem reduces the equation and in the special case of cylindrical membranes it becomes the ordinary differential equation

$$2\frac{\mathrm{d}^2\kappa}{\mathrm{d}s^2} + \kappa^3 - \mu\kappa - \sigma = 0. \tag{3}$$