



## BOOK REVIEW

*Integrable Hamiltonian Hierarchies, Spectral and Geometric Methods*, by Vladimir Gerdjikov, Gaetano Vilasi and Alexandar Yanovski, Springer, Heidelberg 2010, 643 pp., ISBN: 978-3-642-09577-1.

In the past decades there has been significant progress in the theory of the non-linear evolution equations that could be solved through some inverse scattering techniques (soliton equations or completely integrable equations). These equations have many applications to various physical theories (both classical and quantum) such as hydrodynamics, plasma physics, nonlinear optics, solid state physics etc. Their study also boosted the mathematical research in Hamiltonian dynamics and more generally in the theory of the dynamical systems and in the spectral theory of operators. The mentioned progress, together with interesting applications, increased the number of research monographs dedicated to the subject of soliton equations. The variety of approaches used in these monographs reflects the variety of the methods used in the study of the soliton equations. In general however, most of them are dedicated to the development of one particular approach and the interrelations with other approaches and ideas frequently remains obscure.

The book under review is one of the few exceptions. It develops in parallel spectral and geometric methods to the topic and also encompasses many algebraic ideas related to these fields. The first edition has been published in 2008, by the same publisher. The appearance of this soft cover reprint shows that the readers appreciated the integral approach of the book and that the topic itself remains popular.

The book has two parts: I) Spectral Theory of the Recursion Operators and II) Geometry of the Recursion Operators, treating the so-called Recursion Operators (also known as generating operators or  $\Lambda$ -operators) from spectral and geometric viewpoint respectively. The recursion operators are one of the essential tools of the soliton equations theory, in fact their study reveals almost everything related to the soliton equations except the exact solutions. Finding the exact solutions requires other techniques which are also treated in the book. The authors contributed to the development of both aspects of the recursion operator theory, discussed in

the book, so naturally it contains some of their own original results. In particular, V. Gerdjikov was one of the first that started the study of the expansions over the adjoint solutions and the recursion operators related to them applying them to the soliton equations. Together with A. Yanovski they have put the theory for the Zakharov-Shabat auxiliary problem and its generalizations in the gauge-covariant form we know now. As for the geometric ideas, G. Vilasi was one of the first to appreciate the role of Nijenhuis tensors for the integrability of dynamical systems and develop this idea in his works. From his side A. Yanovski contributed significantly to the geometric approach to the recursion operators also, in some sense providing the link between the two approaches. Together with the things we already mentioned, the book also introduces many results that are now classical. Because of the above circumstances the book provides both historical background and essential information about the theory of the soliton equations. Also, the text is practically self-contained which has been one of the goals of the authors. All the above makes the book useful to those that would like to enter the area of soliton theory and to acquire a broad overview on it. This of course carries some risks. Indeed, due to the diversity of the methods used the mathematical sophistication of the book is above the student level, so the target audience consists mainly of the following groups: PhD students who want to study more profoundly the field, and researchers in the field of soliton theory and its applications. Speaking about the second group, even researchers who know very well some of the results will be interested to see how the geometric and analytic theories are brought together. They will also find that the book is quite useful as a source of references. In addition, even when very well known problems are explained, there are number of interesting examples added, some of which deserve to be studied on their own right. Here I will mention the new point of view on the old classical problem of Kepler dynamics (Subsection 14.3.2) and the relation between complex Hamiltonian systems and real Hamiltonian systems (Subsection 12.2).

The book is clearly written and well organized. The discussion about a particular topic usually begins with preliminaries and some historical background. Next, in the description of the general theory all the essential details and computations are explicitly given. Moreover, the text includes many relations and formulae written in details which is quite useful to anyone who needs to reproduce the calculations. On the other hand, the book is useful to those, interested only in particular result, since the most essential equations and notations are indexed and properly referenced so that it is easy to find any necessary previous information. The bibliography given at the end of each chapter is impressive. All in all the book is really reader-friendly. In some more details the content of the book is as follows.

Part I. After an Introduction with historical remarks the first part continues introducing the now famous Ablowitz-Kaup-Newell-Segur (AKNS) approach to the soliton equations. The theory is developed for the case of Zakharov-Shabat auxiliary linear problem  $L\psi = 0$ . Prior to developing the theory, some background information is given. Chapters two to four introduce the reader to several important topics: the spectral theory of the operator  $L$ , direct and Inverse Scattering Method (ISM) as well as various techniques for obtaining exact solutions to the soliton equations, such as the Gelfand-Levitan-Marchenko equation, the Riemann-Hilbert problem and the dressing method. In brief, these are all the essential techniques and methods that those intending to work in the field of soliton equations would need. Section 5 begins with the spectral theory of the recursion operator (generating operator). The notion has been introduced earlier in the book but it appears there again in relation to the so-called *expansions over adjoint solutions* known also as “squared solutions”. In the next Chapter 6 it is demonstrated that all fundamental properties of the soliton equations are generated by the recursion operators. Chapter 7 demonstrates that the soliton equations allow hierarchy of Hamiltonian structures. The effects of the involutions, both on the equations and their Hamiltonian structures, are outlined. In Chapter 8 the authors analyze the gauge transformations as an important tool of obtaining new soliton equations from old ones. In the last Chapter 9 of Part I the notion of the classical  $R$ -matrix as an alternative approach to the Hamiltonian properties of the soliton equations is outlined.

Part II. This part is aimed at explaining the idea that the recursion operators could be interpreted geometrically as mixed  $(1, 1)$  tensor field on the infinite dimensional manifold  $\mathcal{M}$  of the “potential functions” that enter in the auxiliary linear problems for the soliton equations. More specifically, the adjoint to the recursion operator is Nijenhuis tensor fields coupled with some canonical Poisson structure(s), thus making  $\mathcal{M}$  a Poisson-Nijenhuis manifold. The ingredients used in order to produce the above picture include some basic knowledge from classical, symplectic and Poisson geometry, Hamiltonian mechanics and momentum maps related to Lie groups action and the properties of the Nijenhuis tensors. After an Introduction and Chapter 10, these ideas are developed also in Chapters 11,12, and the information about Nijenhuis tensors is given in Chapter 13. These ingredients are brought together in Chapter 14 introducing compatible Poisson structures. Chapters 14 and 15 are essential for the understanding of the geometry of the recursion operators related to the Zakharov-Shabat system and its generalizations in canonical and pole gauge. They clarify the Poisson-Nijenhuis geometry of three manifolds: the manifolds of potentials in different gauges for the generalized Zakharov-Shabat system ( $L$  and  $\tilde{L}$ ) and the manifold of the Jost solutions for the system  $L$ . Chapter 14 also shows that recursion operators are not necessarily related to symplectic structures,

they exist also for non-Hamiltonian systems in which there are sufficient Abelian symmetries ensuring the integrability. Chapter 16 discusses some natural ways of obtaining compatible Poisson structures arising from bundles of Lie algebras and their applications to the study of recursion operators related to the chiral fields equations hierarchy and Landau-Lifshitz equation hierarchy.

The book ends with an Appendix in which some generalizations are described including some algebraic generalizations of the basic notions of dynamical systems theory.

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