



BOOK REVIEW

The Defocusing NLS Equation and Its Normal Form by Benoît Grébert and Thomas Kappeler, European Mathematical Society, Zürich 2014, x+166pp, ISBN: 978-3-03719-131-6.

The authors are famous mathematicians. Professor Thomas Kappeler is a Director of the Institute for Mathematics at University of Zürich, Switzerland. Professor Benoit Grebert is a Director of the Jean Leray Laboratory of Mathematics at the University of Nantes, France.

Their book is dedicated to the analysis of the defocusing nonlinear Schrödinger equation (NLS)

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - 2|u|^2u = 0. \quad (1)$$

The authors claim that the same methods can be applied also to the focusing NLS equations, which is different from (1) only in the sign of the nonlinear term

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2u = 0. \quad (2)$$

In order to relate to each of these equations a valid mathematical problem one needs to specify first the class of functions \mathcal{M} to which the solution $u(x, t)$ should belong. From this point of view there are three different classes of solutions: i) the class of smooth functions \mathcal{M}_1 vanishing fast enough for $x \rightarrow \pm\infty$; ii) the class of smooth functions \mathcal{M}_2 tending fast enough to constant for $x \rightarrow \pm\infty$, i.e. $\lim_{x \rightarrow \pm\infty} u(x, t) = \rho e^{\pm i\phi_0}$; iii) the class of smooth periodic functions \mathcal{M}_3 .

All three types of problems have their specific peculiarities. In a sense, the periodic problem is the most general one; the other two can be obtained as limiting cases of it [1]. It is only natural that the best studied cases are i) and ii) for the focusing NLS. An important reason for that is that it has a number of important physical applications [1, 2].

Both equations have many features in common. First of all, they both allow Lax representation [2, 3]. This means that one can apply to both of them the inverse

scattering method [1–4] which allows one to prove theorem for existence and uniqueness of the solution for each of the classes i) - iii). Both equations possess an infinite number of integrals of motion and can be described as infinite dimensional completely integrable Hamiltonian systems. For the classes i) and ii) in the focusing case one can in addition: a) prove the existence of an infinite number (a hierarchy) of Hamiltonian structures, b) derive their soliton solutions, c) derive their action-angle variables in terms of the scattering data and their variations, d) construct recursion operators which generate the above-mentioned Hamiltonian formulations.

There is also an important difference between the focusing and defocusing NLS equations for the cases i) and ii): A) The Lax operator L_1 for the focusing case (2) is not self-adjoint, while for the defocusing case the Lax operator L_2 is self-adjoint. Probably this last fact has been decisive for the choice of the topic of the book; B) the focusing NLS (2) allows stable soliton solutions whose interactions are purely elastic, while the defocusing NLS (1) has no proper soliton solutions.

All these problems are much more difficult to handle for the class iii) problems, although it seems natural that most of the above-mentioned results should allow generalizations also for the case iii).

My impression is that Grebert and Kappeler had the ambition to study the most difficult case iii) keeping the level of rigor as high as possible. The fact that the operator L_2 is self-adjoint simplifies substantially their task. The book consists of an overview (instead of an Introduction), four chapters and six Appendices.

In the Overview the authors briefly describe topics covered by the book.

Chapter I is entitled “Zakharov-Shabat operators”. Here the fundamental solution of the operator L_2 for the class of periodic potentials $u(x+1, t) = u(x, t)$ are constructed. The authors give rigorous proofs of their existence based on a number of estimations. The chapter ends with analysis of the gradients and discriminants of the problem. As a rule they first treat more general classes of potentials and, respectively, more general NLS equations for two complex functions φ_1, φ_2 that go into the defocusing NLS only after the additional constraint $\varphi_1 = \varphi_2^*$.

In Chapter II “Spectra”, the spectral properties of two types of problems: the Dirichlet and the Neumann one are outlined. They both have a set of infinite number of discrete eigenvalues, that determine the end points of the continuous spectrum of L_2 . Next the canonical Poisson brackets are introduced in the space of allowed potentials and then the isospectral sets in terms of the scattering data of L_2 were introduced.

Chapter III, entitled “Liouville coordinates” is dedicated to the construction of the action-angle coordinates of the defocusing NLS equation (1). The analysis here

requires an excellent knowledge of the elliptic and hyper-elliptic functions. First potentials of almost real type are considered. Then the set of action-type variables are constructed and shown to be in involution. Next the angle-type variables, determined by the Psi-functions are introduced. It requires substantial efforts to prove that the so-introduced action-angle variables satisfy canonical Poisson brackets and demonstrate rigorously that they have all the properties they must have. One should stress here the fact, that the authors are dealing with infinite-dimensional Hamiltonian systems which puts additional obstacles in all considerations.

The last Chapter IV “Birkhoff coordinates” is devoted to the derivation of the Birkhoff coordinates, which are at the same time, real analytic functions of the action-angle variables. Their Jacobians, normal forms and diffeomorphism properties are thoroughly analyzed.

The Appendices contain some supplementary material.

In all the authors have achieved their goal to construct rigorously the action-angle variables for the defocusing NLS equation (1). However the analysis of the Hamiltonian hierarchy at this level of rigor, or constructing the recursion operators for case iii) still remain a challenge. Also the list of relevant references could be extended.

References

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