SOME REMARKS ON THE EXPONENTIAL MAP ON THE GROUPS $\text{SO}(n)$ AND $\text{SE}(n)$

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Abstract. The problem of describing or determining the image of the exponential map $\exp : \mathfrak{g} \rightarrow G$ of a Lie group $G$ is important and it has many applications. If the group $G$ is compact, then it is well-known that the exponential map is surjective, hence the exponential image is $G$. In this case the problem is reduced to the computation of the exponential and the formulas strongly depend on the group $G$. In this paper we discuss the generalization of Rodrigues formulas for computing the exponential map of the special orthogonal group $\text{SO}(n)$, which is compact, and of the special Euclidean group $\text{SE}(n)$, which is not compact but its exponential map is surjective, in the case $n \geq 4$.

1. Introduction. Lie Groups and the Exponential Map

Let $G$ be a Lie group with its Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \rightarrow G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in \mathfrak{g}$ and $\gamma_X$ is the one-parameter subgroup of $G$ induced by $X$. Recall the following general properties of the exponential map:

1. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(tX) = \gamma_X(t)$
2. For every $s, t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have
   \[ \exp(sX) \exp(tX) = \exp(s + t)X \]
3. For every $t \in \mathbb{R}$ and for every $X \in \mathfrak{g}$, we have $\exp(-tX) = \exp(tX)^{-1}$
4. $\exp : \mathfrak{g} \rightarrow G$ is a smooth mapping, it is a local diffeomorphism at $0 \in \mathfrak{g}$ and $\exp(0) = e$, where $e$ is the unity element of the group $G$
5. The image $\exp(\mathfrak{g})$ of the exponential map generates the connected component $G_e$ of the unity $e \in G$