BERTRAND SYSTEMS ON SPACES OF CONSTANT
SECTIONAL CURVATURE. THE ACTION-ANGLE ANALYSIS.
CLASSICAL, QUASI-CLASSICAL AND QUANTUM PROBLEMS

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Abstract. Studied is the problem of degeneracy of mechanical systems the
configuration space of which is the three-dimensional sphere, the elliptic
space, i.e., the quotient of that sphere modulo the antipodal identification,
and finally, the three-dimensional pseudo-sphere, namely, the Lobatchevski
space. In other words, discussed are systems on groups SU(2), SO(3, \mathbb{R}), and
SL(2, \mathbb{R}) or its quotient SO(1, 2). The main subject are completely degener-
ate Bertrand-like systems. We present the action-angle classical description,
the corresponding quasi-classical analysis and the rigorous quantum formu-
las. It is interesting that both the classical action-angle formulas and the rig-
gorous quantum mechanical energy levels are superpositions of the flat-space
expression, with those describing free geodetic motion on groups.

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sphere

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