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QUANTUM INTEGRABILITY AND COMPLETE SEPARATION OF VARIABLES FOR PROJECTIVELY EQUIVALENT METRICS ON THE TORUS

VLADIMIR S. MATVEEV

Isaac Newton Institute, Cambridge CB3 0EH, UK and

Department of Algebra and Geometry, State University, 454021 Chelyabinsk, Russia

Abstract. Let two Riemannian metrics g and $\bar{g}$ on the torus T^n have the same geodesics (considered as unparameterized curves). Then we can construct invariantly n commuting differential operators of second order. The Laplacian Δ_g of the metric g is one of these operators. For any $x \in T^n$, consider the linear transformation G of T_xT^n given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. If all eigenvalues of G are different at one point of the torus then they are different at every point; the operators are linearly independent and we can globally separate the variables in the equation $\Delta_g f = \mu f$ on this torus.

1. Commuting Operators for Projectively Equivalent Metrics

Let g and $\bar{g}$ are two C^2-smooth Riemannian metrics on some manifold M^n. They are **projectively equivalent** if they have the same geodesics considered as unparameterized curves.

The problem of describing projectively equivalent metrics was stated by Beltrami in [1]. Locally, in the neighborhood of so-called sable points, it was essentially solved by Dini [3] for surfaces and by Levi-Civita [4] for manifolds of arbitrary dimension. Denote by G the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. In invariant terms, G is the fiberwise-linear mapping $G\colon TM^n \to TM^n$ such that its restriction to any tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ satisfying the following condition: for any vectors $\xi, \nu \in T_{x_0}M^n$, the scalar product $g(G(\xi), \nu)$ of the vectors $G(\xi)$ and ν in g is equal to the scalar product $\bar{g}(\xi, \nu)$ of the vectors ξ and ν in $\bar{g}$.