THE CANHAM-HELFRICHT MODEL FOR THE ELASTICITY OF BIOMEMBRANES AS A LIMIT OF MESOSCOPIC ENERGIES

LUCA LUSSARDI

Dipartimento di Scienze Matematiche “G.L. Lagrange”, Politecnico di Torino
10129 Torino, Italy

Abstract. In this paper we review some recent results concerning the variational deduction of a Canham-Helfrich model for biomembranes obtained starting from a mesoscopic model which implements the amphiphilic behavior of the lipid molecules and the head-tail connection. The two-dimensional analysis is complete while in the three-dimensional case we have partial results and open problems.

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1. Introduction

A prominent way to model biomembranes is given by shape energies of Canham-Helfrich type \([1,3,4,7,17]\). These type of energies have the general form

\[
E(S) = \int_S \kappa_1 (H - H_0)^2 - \kappa_2 K \, dH^2
\]

where \( S \) denotes a smooth surface in \( \mathbb{R}^3 \), \( H \) and \( K \) are the mean curvature and the Gaussian curvature of \( S \) respectively, and the bending moduli \( \kappa_1, \kappa_2 \) and the spontaneous curvature \( H_0 \) are constant. Typically, \( \kappa_1 > \kappa_2 > 0 \) is a compatibility condition coming both from mathematical considerations and from experiments \([14,16]\). The shape of the membrane is an absolute minimizer of \( E \) among a suitable class of surfaces. We notice that, thanks to the Gauss-Bonnet’s Theorem, when the spontaneous curvature is zero and the topology of \( S \) is fixed the minimization problem for the Canham-Helfrich functional reduces to the minimization problem for the very well studied Willmore functional \([9,13,15]\). The